Mathematical Notation

Math 230 - Differential Equations

Name : ____________________________

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Include the title as part of what you type. The lines around the title aren't that important, but if you will type ----- at the beginning of a line and hit enter, both Word and WordPerfect will draw a line across the page for you.

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For individual symbols (μ, σ, etc), you can insert symbols. In Word, use "Insert / Symbol" and choose the Symbol font. For WordPerfect, use Ctrl-W and choose the Greek set. For more complex expressions you should use the equation editor. If there is an equation, put both sides of the equation into the same equation editor box. You will run into trouble in chapter 7, see the notes on the Internet for help.

There are instructions on how to use the equation editor in a separate document. Be sure to read through the help it provides. There are some examples at the end that walk students through the more difficult problems. You will want to read the handout on using the equation editor if you have not used this software before.

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These notations are due two class periods after we finish the material for that chapter. See the calendar for exact due dates. Late work will be accepted but will lose 10% per class period.

Chapter 1 - Introduction to Differential Equations

Theorem 1.1: A differential equation will have an unique solution if both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on some region.

Mathematical Models

Population Dynamics: The rate of population growth is proportional to total population at that time. $\frac{dP}{dt} = kP$

Radioactive Decay: The rate at which the nuclei of a substance decay is proportional to the number of nuclei remaining. $\frac{dA}{dt} = kA$

Newton's Law of Cooling: The rate at which the temperature a body changes is proportional to the difference between the temperature of the body and the surrounding medium. $\frac{dT}{dt} = k(T - T_m)$

Chemical Reactions: The rate at which a reaction proceeds is proportional to the product of the remaining concentrations. $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$

Series Circuits: Kirchoff's second law says $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

Falling Bodies: Without air resistance and a positive upwards direction, $m\frac{d^2s}{dt^2} = -mg$ or $m\frac{dv}{dt} = -mg$. With air resistance (viscous damping) and a positive negative direction, $m\frac{dv}{dt} = mg - kv$ or $m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg$.

Slipping Chain: For a chain in motion around and frictionless peg, $\frac{d^2x}{dt^2} - \frac{2g}{L}x = 0$

Suspended Cables: If $\mathbf{T}_1$ is the tension tangent to the lowest point and $\mathbf{W}$ is the portion of the vertical load between two points, then $\frac{dy}{dx} = \frac{\mathbf{W}}{\mathbf{T}_1}$

Chapter 2 - First-Order Differential Equations

A first-order DE is separable if it can be written in the form $\frac{dy}{dx} = g(x)h(y)$

The standard form for a linear first-order DE is $\frac{dy}{dx} + P(x)y = f(x)$ and is homogeneous if $f(x) = 0$. The solution to this DE is the sum of two solutions $y = y_c + y_p$ where y_c is the general solution to the homogenous DE and y_p is the particular solution to the nonhomogeneous DE. The procedure known as variation of parameters leads to an integrating factor $u = e^{\int P(x)dx}$.

The error function and complementary error functions are defined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ and $erf(x) + erfc(x) = 1$.

For a function $z = f(x, y)$, the differential $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$. If the function is a constant, then the differential is 0. A DE of the form $M(x, y)dx + N(x, y)dy = 0$ is an exact differential equation if the left hand side is a differential of some function $f(x, y)$. If M and N are continuous and have continuous partial derivatives on some region, then it is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If a DE is exact, then you can find the potential function $f(x, y)$ by integrating $\int Mdx$ and $\int Ndy$ and finding the union of all the terms.

A function is homogeneous of degree α if it has the property that $f(tx, ty) = t^\alpha f(x, y)$. The substitutions $y = ux$ or $x = vy$ will reduce a homogenous equation to a separable first-order DE.

Bernoulli's equation is $\frac{dy}{dx} + P(x)y = f(x)y^n$ and can be solved with the substitution $u = y^{1-n}$.

Chapter 3 - Modeling with First-Order Differential Equations

Kirchoff's Laws:
Let $E(t)$ be impressed voltage, $i(t)$ be current, $q(t)$ be charge, L be inductance, R be resistance, and C be capacitance. Current and charge related by $i(t)=\frac{dq}{dt}$.

Conservation of Charge (1st law): The sum of the currents entering a node must equal the sum of the currents exiting a node.

Conservation of Energy (2nd law): The voltages around a closed path in a circuit must sum to zero (voltage drops are negative, voltage gains are positive). The voltage drop across an inductor is $L\frac{di}{dt}=L\frac{d^2q}{dt^2}$. The voltage drop across a resistor is $iR=R\frac{dq}{dt}$. The voltage drop across a capacitor is $\frac{1}{C}q$. The sum of the voltage drops is equal to the impressed voltage $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$.

Logistic Equation: When the rate of growth is proportional to the amount present and the amount remaining before reaching the carrying capacity K, then the resulting DE is $\frac{dP}{dt}=P(a-bP)$ and the solution is $P(t)=\frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$

Lotka-Volterra Predator-Prey Model: If $x(t)$ is the population of a predator and $y(t)$ is the population of the prey at time *t*, then the populations can be modeled by the system of nonlinear system of DEs: $\frac{dx}{dt}=x(-a+by)$ and $\frac{dy}{dt}=y(d-cx)$

Chapter 4 - Higher-Order Differential Equations

Superposition Principle - Homogeneous Equations: A linear combination of solutions to a homogeneous DE is also a solution. This means that constant multiples of a solution to a homogeneous DE are also solutions and the trivial solution $y = 0$ is always a solution to a homogeneous DE.

A set of functions is linearly dependent if there is some linear combination of the functions that is zero for every *x* in the interval.

A set of solutions is linearly independent if and only if the Wronskian is not zero for every *x* in some interval. A set of linearly independent solutions to a homogeneous DE is set to be a fundamental set of solutions and there is always a fundamental set for a homogeneous DE. The Wronskian is defined by

$$W\left(f_1, f_2, \cdots, f_n\right) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Any function free of arbitrary parameters that satisfies a nonhomogeneous DE is a particular solution, y_p. The complementary function, y_c, is the general solution to the associated homogeneous DE. The general solution to a nonhomogeneous equation is $y = y_c + y_p$

Reduction of Order: If $y_1(x)$ is a solution to a second-order linear homogeneous DE in standard form $y'' + P(x)y' + Q(x)y = 0$, then a second solution is

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

Homogeneous Linear Equations with Constant Coefficients: The auxiliary equation is formed by converting the DE into a polynomial function. For example, $3y^{(5)} - 20y^{(4)} + 78y''' - 134y'' + 99y' - 26y = 0$ would have an auxiliary equation of $3m^5 - 20m^4 + 78m^3 - 134m^2 + 99m - 26 = 0$. Find the solutions to the auxiliary equation, which in this case are $m = 1$ with multiplicity 2, $m = 2/3$, and $m = 2 \pm 3i$. From each of the roots, we form a linear independent combination of terms involving *e*. Thus $y = c_1 e^x + c_2 x e^x + c_3 e^{\frac{2}{3}x} + e^{2x}\left(c_4 \cos 3x + c_5 \sin 3x\right)$. Two common DEs $y'' + k^2 y = 0$ and $y'' - k^2 y = 0$ have solutions of $y = c_1 \cos kx + c_2 \sin kx$ and $y = c_1 e^{kx} + c_2 e^{-kx}$ respectively. The solutions to $y'' - k^2 y = 0$ can also be written as $y = c_1 \cosh x + c_2 \sinh x$.

Method of Undetermined Coefficients - Superposition Approach: This method is useful when the coefficients of the DE are constants and the input function is comprised of sums or products of constant, polynomial, exponential, or trigonometric (sine and cosine) functions. You make guesses about the particular solutions based on the form of the input and then equate coefficients.

Method of Undetermined Coefficients - Annihilator Approach: *L* is an annihilator of a function if it has constant coefficients and $L\left(f\left(x\right)\right) = 0$. Use D^n to annihilate functions of the form x^k. Use $\left(D - \alpha\right)^n$ to annihilate functions of the form $x^k e^{\alpha x}$. Use $\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]^n$ to annihilate functions of the form $x^k e^{\alpha x} \cos \beta x$ or $x^k e^{\alpha x} \sin \beta x$. In each case, the *k* is a whole number less than *n*.

Variation of Parameters: Variation of parameters can be used when the coefficients of the DE are not constants. It involves the Wronskian and two functions $u_1{}' = -\frac{y_2 f\left(x\right)}{W}$ and $u_2{}' = -\frac{y_1 f\left(x\right)}{W}$ that are integrated to find u_1 and u_2. The particular solution is then $y_p = u_1 y_1 + u_2 y_2$.

Cauchy-Euler Equation: A linear differential equation composed of terms $a_k x^k \frac{d^k y}{dy^k}$, where the a_k factors are constant, can be solved by trying $y = x^m$. Treat like the auxiliary equation, except use $\ln x$ instead of *x*. For example, if the solutios are $m = 2 \pm 3i$, then $y = e^{2\ln x}\left[c_1 \cos\left(3\ln x\right) + c_2 \sin\left(3\ln x\right)\right]$, which simplifies to $y = x^2\left[c_1 \cos\left(3\ln x\right) + c_2 \sin\left(3\ln x\right)\right]$.

Chapter 5 - Modeling with Higher-Order Differential Equations

Free Undamped Motion: $m\frac{d^2x}{dt^2} = -kx$ can be written as $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$ and has a solution of $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

Free Damped Motion: $m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$ can be written as $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$ where $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$. If $\lambda^2 - \omega^2 > 0$, the system is overdamped and $x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right)$. If $\lambda^2 - \omega^2 = 0$, the system is critically damped and the solution is $x(t) = e^{-\lambda t}(c_1 + c_2 t)$. If $\lambda^2 - \omega^2 < 0$, the system is underdamped and the solution is $x(t) = e^{-\lambda t}\left[c_1 \cos\left(t\sqrt{\omega^2-\lambda^2}\right) + c_2 \sin\left(t\sqrt{\omega^2-\lambda^2}\right)\right]$.

Driven Motion: In driven motion, an external force $f(t)$ is applied to the system and the DE is $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$ where $F(t) = \frac{f(t)}{m}$. Use the method of undetermined coefficients or variation of parameters to solve the nonhomogeneous equation.

Series Circuit Analogue: The DE $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ is overdamped, critically damped, or underdamped depending on the value of the discriminant $R^2 - 4L/C$.

Deflection of a Beam: Deflection satisfies the DE $EI\frac{d^4y}{dx^4} = w(x)$ where EI is the flexural rigidity and $w(x)$ is the load per unit length.

Chapter 6 - Series Solutions of Linear Equations

If $x = x_0$ is an ordinary point, then a power series centered at x_0 is

$$y = \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^k .$$

Method of Frobenius: If $x = x_0$ is a regular singular point then there exists at least one solution of the form $y = \left(x - x_0\right)^r \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^k$ which simplifies to

$y = \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^{k+r}$, where *r* is a constant to be determined.

Bessel's Equation of Order *v*: $x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0$

Bessel Functions of the First Kind: The two functions are

$J_v\left(x\right) = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k}{k!\Gamma\left(1+v+k\right)} \left(\frac{x}{2}\right)^{2k+v}$, which converges on $\left[0,\infty\right)$ if $v \geq 0$, and

$J_{-v}\left(x\right) = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k}{k!\Gamma\left(1-v+k\right)} \left(\frac{x}{2}\right)^{2k-v}$, which converges on $\left(0,\infty\right)$. $J_v\left(x\right)$ and $J_{-v}\left(x\right)$ are linearly independent, so the general solution of Bessel's Equation of order *v* is $y = c_1 J_v\left(x\right) + c_2 J_{-v}\left(x\right)$ where *v* is not an integer.

Bessel Functions of the Second Kind: If *v* is not an integer, then $J_v\left(x\right)$ and $Y_v\left(x\right) = \frac{\cos v\pi J_v\left(x\right) - J_{-v}\left(x\right)}{\sin v\pi}$ are linearly independent solutions of the Bessel's equation of order *v* so the general solution can be written as $y = c_1 J_v\left(x\right) + c_2 Y_v\left(x\right)$

Legendre's Equation of Order *n*: $\left(1 - x^2\right) y'' - 2xy' + n\left(n+1\right) y = 0$. If $P_n\left(x\right)$ is the solution for order *n*, then some of the solutions are $P_0\left(x\right) = 1$, $P_1\left(x\right) = x$, $P_2\left(x\right) = \frac{1}{2}\left(3x^2 - 1\right)$, $P_3\left(x\right) = \frac{1}{2}\left(5x^3 - 3x\right)$, and $P_4\left(x\right) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$.

Chapter 7 - The Laplace Transform

Let f be a function defined for $t \geq 0$. The Laplace transform of $f(t)$ is $\mathfrak{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, provided this integral converges. $F(s) = \mathfrak{L}\{f(t)\}$.

Laplace transform of a derivative:
$\mathfrak{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$.

First Translation Theorem: $\mathfrak{L}\left\{e^{at} f(t)\right\} = F(s-a)$

Unit Step Function: Also known as the Heaviside function, it is useful for creating piecewise functions. $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$

Second Translation Theorem: If $a > 0$ then $\mathfrak{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as} F(s)$

Derivatives of Transforms: $\mathfrak{L}\left\{t^n f(t)\right\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$

Convolution: Convolutions, defined by $f * g = \int_0^t f(\tau)\, g(t-\tau)\,d\tau$ are commutative, $f * g = g * f$, and the Laplace transform of a convolution is the product of the Laplace transforms, $\mathfrak{L}\{f * g\} = \mathfrak{L}\{f(t)\}\mathfrak{L}\{g(t)\} = F(s)G(s)$. If you let $g(t) = 1$, then the transform of an integral is $\mathfrak{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s}$.

Transform of a Periodic Function: $\mathfrak{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$, where the period is T.

Dirac Delta Function: $\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)$ is ∞ when $t = t_0$ and 0 otherwise. $\int_0^\infty \delta(t-t_0)\,dt = 1$. $\mathfrak{L}\{\delta(t-t_0)\} = e^{-st_0}$

Chapter 8 - Systems of Linear First-Order DEs

Eigenvalues and Eigenvectors: If $\mathbf{X}' = \mathbf{AX}$ is a homogeneous linear first-order system, then the polynomial equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is the characteristic equation and it's solutions are the eigenvalues. We want to write a solution as $\mathbf{X} = \mathbf{K}e^{\lambda t}$ where $\mathbf{K}$ is the associated eigenvector.

The general solution to a homogenous solution is
$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}$

If your solutions correspond to a complex eigenvalue $\lambda = \alpha + \beta i$, then $\mathbf{K}e^{\lambda t}$ and $\bar{\mathbf{K}}e^{\bar{\lambda} t}$ are both solutions.

For a nonhomogeneous system, the general solution becomes $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ and the method of undetermined coefficients or variation of parameters can be used to find the particular solution.

Matrix Exponentials: For a homogeneous system, we can define a matrix exponential $e^{\mathbf{A}t}$ so that $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution to $\mathbf{X}' = \mathbf{AX}$. For any square matrix of size *n*, $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \dfrac{t^2}{2!} + \cdots + \mathbf{A}^n \dfrac{t^n}{n!}$, which can be written as $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \dfrac{t^k}{k!}$. $e^{\mathbf{A}t}$ is a fundamental matrix.

For nonhomogeneous systems, $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, the general solution is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds$. In practice, $e^{-\mathbf{A}s}$ can be found from $e^{\mathbf{A}t}$ by substituting $t = -s$.

Chapter 9 - Numerical Solutions of Ordinary DEs

Euler's Method: In chapter 2 (and in Calculus II), we had Euler's Method, where $y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Improved Euler's Method: This method estimates the next *y* value in the sequence using Euler's method, $y^*_{n+1} = y_n + h \cdot f(x_n, y_n)$, and then uses that estimate in a midpoint formula to find the next *y* used.

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$$

Runge-Kutta Methods: These are generalizations of Euler's method where the slope $f(x_n, y_n)$ is replaced by a weighted average of the slopes on the interval $x_n \le x \le x_{n+1}$. That is, $y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + \cdots + w_mk_m)$ where the weights *w* are chosen so that they agree with a Taylor series of order *m*.

RK1: The first-order Runge-Kutta method is actually Euler's method. Choose $k_1 = f(x_n, y_n)$ and $w_1 = 1$ to get $y_{n+1} = y_n + h \cdot f(x_n, y_n)$.

RK2: The second-order Runge-Kutta method chooses values $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + h, y_n + hk_1)$, and $w_1 = w_2 = \frac{1}{2}$ to get the improved Euler's method where $y_{n+1} = y_n + h\left(\frac{1}{2} f(x_n, y_n) + \frac{1}{2} f(x_n + h, y_n + hk_1)\right)$

RK4: Let $w_1 = w_4 = \frac{1}{6}$ and $w_2 = w_3 = \frac{1}{3}$. Choose $k_1 = f(x_n, y_n)$, $k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$, $k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right)$, and $k_4 = f(x_n + h, y_n + hk_3)$.