chapter 4 (Probability Distributions)

Lecture:
There are two types of random variables: Discrete and continuous. Suppose you flip a coin 3 times; let X be the number of heads. Then, the possibility would be X = 0, 1, 2, 3 (that is, you would see heads none, one, two, or three times). We call X a random variable. So, a random variable is a variable that takes on numerical value for each possible outcome of a probability experiment. When the RV is used for counting (usually X is 0, 1, 2, 3,...), then X is called a discrete RV. For example suppose you have 2 white balls and 8 blue balls mixed; and you randomly select 2 balls. What are the possibilities for the white balls? Or, how many white balls are possible to be seen in the sample you took? The answer is X = 0, 1, 2. A continuous RV is a variable that can assume infinitely many values from a continuous interval (there is no gaps between possible values, for example between 1 and 2 there are infinite numbers). Weights, measurements are examples of CRV's. In chapter 5 and the following chapters you will see CRV's. For a discrete probability distribution (DPD) two conditions must be met: 1. Probabilities for the RV must be between zero and one. 2. The sum of the probabilities for all X must be exactly one. Symbolically they are : \(0 \leq P(X) \leq 1\), \(\Sigma P(X) = 1\).

Formulas for the mean and variance of a DPD: \(\mu = \Sigma X P(X) = E(X)\), \(\text{Var} = \sigma^2 = \Sigma X^2 P(X) - \mu^2\).

A binomial experiment includes the following conditions and notations:
- the experiment is repeated n times (n is called the number of trials).
- there are only two outcomes (success (S) and failure (F)).
- the probability of success is named p and the probability of failure is named q.
- the trials are independent of each other (knowing what happened before does not change p).
- p is a fixed number for each trial (this is a rehash of the previous condition).
- x is the number success for the experiment: x = 0, 1, 2,...,n.
- It should be clear that \(P(S)+P(F)=p+q=1\) (since F complements S).

The best example of a binomial experiment is flipping a coin n times (note that all the conditions are met).

The formula is:
\[p(X) = \binom{n}{X} \, p^X \, q^{n-X}\], where \(\binom{n}{X}\) is another notation for combinations.

Example 1: Suppose a drug for headaches work on 40% of headache sufferers. If we randomly select 10 patients with headaches and give them this drug what is the probability that exactly 3 of them rid of their headaches?

Solution 1: In this example \(n = 10\), \(x = 3\), \(p = .4\), and \(q = .6\).
All we have to do is plug in these numbers in the formula.
\[ p(3) = \binom{10}{3} (0.4)^3 (0.6)^7, \text{ note that } \binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 10 \cdot 3 \cdot 4 = 120 \]

\[ p(3) = (120)(0.4)^3 (0.6)^7 = 0.2150 \]

If you wanted to find the probability that at least 3 patients rid of their headaches, then the question symbolically would be: \( p(X \geq 3) \). And to answer this question you must find the following probabilities. \( p(X \geq 3) = p(X = 3) + p(X = 4) + \cdots + p(X = 10) \).

Or you could have found the complement it and subtract it from one as follows:

\[ p(X \geq 3) = 1 - p(X \leq 2). \]

\textit{NOTE:} Sometimes we can reduce a PD, which is not a BPD to a BPD. For example, selecting cards with replacement is a BPD. If you are interested in aces, then success is seeing an ace and failure would be not seeing an ace.

\textit{NOTE:} The binomial distribution is the most important discrete probability distribution.

The other discrete prob. distr's. that we study in this course are: \textbf{Geometric distribution}, \textbf{Poisson distribution}, and \textbf{hypergeometric distribution}.

For a \textbf{geometric distribution} there are only two outcomes (S and F); trials are independent of each other; \( P(S) \) is the same for each trial. For the GD we define \( X \) to be the number of times we do an experiment before we see a success. In a geometric probability distribution we are interested in finding the probability of seeing the first success in \( X \) trials. Russian roulette is a good example. Other examples are questions like: In flipping a coin, what is the probability that the first H will happen in the 3rd trial? It is very easy to answer this question without knowing what GD or its formula is. Use simple logic in probability: The pattern is FFS, therefore \((1/2)(1/2)(1/2) = 1/8\). The second example would be rolling a die. The question would be: What is the probability of seeing a 6 in the 4th trial. The answer, again, is very simple: The pattern is FFFS (remember the not seeing a six is a failure) so, \( p = (5/6)(5/6)(5/6)(1/6) \).

In a \textbf{hypergeometric distribution} the population is finite and the trials are not independent of each other. Usually the situations are like these: Suppose we have 3 red (R) balls and 5 blue (B) balls. We randomly select 3 balls (without replacement); find the probability of seeing one R and two B balls? Since I can not show the solution mathematically, then I will explain the solution: Combination of 3 & 1 (this takes care of the red ball in you sample) times combination of 5 & 2 (this takes care of the blue balls in your sample) divided by combination of 8 & 3 (this takes care of number of different ways you select 3 balls among 8 ball). \( p = (3)(10)/56 \). To simplify it: \( p = 15/28 \).

\textit{NOTE:} With probabilities, the final answer should be a reduced fraction.

When we are concern with things like: Number of defects in one square foot of a fabric, number of errors I have on one page, number of telephone calls I get on a given day, or... we use a \textbf{Poisson distribution}. One important condition is that for example, the number of errors I have one page is independent of the other pages. Example: The mean number of defects in a fabric is 5 per square yard. You buy one yard of this fabric. What is the probability that there are only 2 defects?
\[ p(x) = \frac{\mu^x e^{-\mu}}{x!}, \quad p(2) = \frac{5^2 e^{-5}}{2!} = 0.0842 \]

The number \( e \), which is called the natural number and was discovered by Euler, is approximately equal to 2.72.

**NOTE**: In a binomial distribution, when \( n \) is large and \( p \) is small (these are the cases that it is very difficult to apply a binomial distribution) one should use a Poisson distribution to approximate a binomial distribution.