Fourier Series

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Let \( f(t) \) be a piecewise continuous function of period \( 2\pi \) which is defined for all \( t \). Then the Fourier Series of \( f(t) \) is:

\[
f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right).
\]

Here we have 3 constants: \( a_0, a_n, b_n \).

Where

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt
\]

*Note that \( a_0 = 0 \) if \( f(t) \) is an odd function.*

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad n = 1, 2, 3, \ldots
\]

*Note that \( a_n = 0 \) if \( f(t) \) is an odd function.*

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt, \quad n = 1, 2, 3, \ldots
\]

*Note that \( b_n = 0 \) if \( f(t) \) is an even function.*

The above equations are Euler formulas.

In order to find \( a_0 \) do the following integration and solve for \( a_0 \).

\[
\int_{-\pi}^{\pi} f(t) \, dt = \int_{-\pi}^{\pi} a_0 \, dt + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \left( a_m \cos mt + b_m \sin mt \right) \, dt
\]

\[
\int_{-\pi}^{\pi} f(t) \, dt = 2\pi a_0 \implies a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt
\]

*Note that \( \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} \left( a_m \cos mt + b_m \sin mt \right) \, dt = 0 \), since the net signed area = 0.*

In order to find \( a_n \) multiply both sides of the series by \( \cos nt \), then integrate both sides with respect to \( t \) and solve for \( a_n \).

\[
\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \int_{-\pi}^{\pi} a_0 \cos nt \, dt + \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} \left( a_m \cos nt \cos mt + b_m \cos nt \sin mt \right) \, dt
\]

\[
\int_{-\pi}^{\pi} f(t) \cos nt \, dt = a_n \int_{-\pi}^{\pi} \cos^2 nt \, dt = \pi a_n
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt.
\]

*Note that \( \int_{-\pi}^{\pi} \frac{a_0}{2} \cos nt \, dt = 0 \) and \( \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} \left( a_m \cos nt \cos mt + b_m \cos nt \sin mt \right) \, dt = \pi a_n \).*

I would like to show that \( \sum_{m=1}^{\infty} \int_{-\pi}^{\pi} \left( a_m \cos nt \cos mt + b_m \cos nt \sin mt \right) \, dt = \pi a_n \).

We know that the terms with the \( b_m \) coefficients are zero (since they are odd functions and \( t_1 = -t_2 \) for all terms).
Let's look at the term with the $a_n$ coefficient.

$$
\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \int_{-\pi}^{\pi} \frac{1}{2} \left[ \cos((m+n)t) + \cos((m-n)t) \right] \, dt
$$

$$
= \frac{1}{2} \left[ \left( \frac{1}{m+n} \right) \sin((m+n)t) + \left( \frac{1}{m-n} \right) \sin((m-n)t) \right]_{-\pi}^{\pi} = \frac{1}{m+n} \sin \pi (m+n) + \frac{1}{m-n} \sin \pi (m-n)
$$

If $m \neq n$, then $\sin k\pi = 0$, $k$ = any integer.

If $m = n$, then $\frac{1}{m-n} \sin \pi (m-n) = 0$; apply La'Hopital's rule we will have

$$
\lim_{m \to n} \frac{1}{m-n} \sin \pi (m-n) = \lim_{m \to n} \pi \cos \pi (m-n) = \pi. \text{ You could have let } z = m-n \text{ and } \lim_{z \to 0} \frac{\sin \pi z}{z} = \pi.
$$

Note: If you don't want to apply La'Hopital's rule, then you can do the following:

$$
\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \int_{-\pi}^{\pi} \cos^2 nt \, dt = \int_{0}^{\pi} (\cos 2nt + 1) \, dt = \left[ \frac{1}{2n} \sin 2nt + t \right]_{0}^{\pi} = \pi
$$

In order to find $b_n$ do the same thing as you did for $a_n$, except use the $\sin nt$.

Some useful identities:

$$
\begin{align*}
\int_{-\pi}^{\pi} \cos mt \cos nt \, dt &= \begin{cases} 
0, & m \neq n \\
\pi, & m = n
\end{cases} \\
\int_{-\pi}^{\pi} \sin mt \sin nt \, dt &= \begin{cases} 
0, & m \neq n \\
\pi, & m = n
\end{cases} \\
\int_{-\pi}^{\pi} \cos mt \sin nt \, dt &= \begin{cases} 
0, & m \neq n \\
0, & m = n
\end{cases}
\end{align*}
$$

Note: You need the following trigonometric identities for evaluating the above integrals. I have shown you the first integral.

$$
\cos mt \cos nt = \frac{1}{2} \left[ \cos((m+n)t) + \cos((m-n)t) \right]
$$

$$
\sin mt \sin nt = \frac{1}{2} \left[ \cos((m-n)t) - \cos((m+n)t) \right]
$$

$$
\cos mt \sin nt = \frac{1}{2} \left[ \sin((m+n)t) - \sin((m-n)t) \right]
$$

If the period is $2L$ instead of $2\pi$, then the Fourier series of $f(t)$ is

$$
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t \right)
$$

Note: If the period of $f$ is $2\pi$ and the period of $g$ is $2L$, then $g(t) = f\left( \frac{\pi}{L} t \right)$. You can check that the period of $g$ is $2L \left( \frac{2\pi}{\pi/L} = 2L \right)$.

Where
\[ a_0 = \frac{1}{L} \int_{-L}^{L} f(t) \, dt \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \]

If the interval is of this form \( 0 < t < 2L \), then it is easier to find the coefficients by using the following equations:

\[ a_0 = \frac{1}{L} \int_{0}^{2L} f(t) \, dt \]

\[ a_n = \frac{1}{L} \int_{0}^{2L} f(t) \cos \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \]

\[ b_n = \frac{1}{L} \int_{0}^{2L} f(t) \sin \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \]

If \( f(t) \) is given for half-period \( L \), then for \( -L < t < 0 \) you have a choice of making \( f(t) \) an even function, \( f(-t) = f(t) \), or an odd function, \( f(-t) = -f(t) \).

If you select \( f(t) \) be an even function then,

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} t\right) \quad \text{with} \quad a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad b_n = 0. \]

If you select \( f(t) \) be an odd function then,

\[ f(t) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi}{L} t\right) \quad \text{with} \quad b_n = \frac{2}{L} \int_{0}^{L} f(t) \sin \frac{n\pi}{L} \, dt, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad a_n = 0, \quad n = 1, 2, \ldots \]

Convergence of Fourier Series-

If \( f \) is a piecewise smooth function, then the Fourier series will converge

a. to \( f(t) \) at every point that \( f \) is continuous,

b. to the average (mean) value of \( f \) at the point of discontinuity. Usually, the function is not defined at \( t = -L, 0, L \). We make \( f \) to be defined at \( t = L \) by

assigning \( f(L) = \frac{\lim_{t \to +L} f(t) + \lim_{t \to -L} f(t)}{2} \). The same applies to all undefined values of \( t \).

The above equation in Fourier notation is presented as: \( f(L) = \frac{f(L^-) + f(L^+)}{2} \).

In many applications of Fourier Series the function is a polynomial. As you know, it takes some time to integrate the product of a polynomial with sine or cosine. I conjectured the following formula. Check the correctness of the following finite series.

Let \( f(t) = t^N, \quad N = 0, 1, 2, \ldots \). We would like to evaluate the following integral:

\[ \int_{-L}^{L} t^N \cos nt \, dt = \sin n \sum_{k=0}^{N/2} \frac{(-1)^k}{n^{2k+1}} f^{(2k)\prime}(t) + \cos n \sum_{k=0}^{N/2} \frac{(-1)^k}{n^{2k+2}} f^{(2k+1)\prime}(t) \]

Here, I have ignored the constant of integration.
\[ \int t^N \sin nt \, dt = \cos nt \sum_{k=0}^{\frac{N}{2}} \frac{(-1)^k}{n^{2k+1}} f^{(2k)}(t) + \sin nt \sum_{k=0}^{\frac{N}{2}} \frac{(-1)^k}{n^{2k+2}} f^{(2k+1)}(t). \]

It is good to remember that \( t^N \) is even when \( N \) is even; in this case \( \int_{-\pi}^{\pi} t^N \sin nt \, dt = 0. \)

It is also good to know that if \( h(t) = f(t) + g(t) \), then the Fourier series of the sum is the sum of the Fourier of \( f \) and \( g \).

Some popular series:

Leibniz’ series: \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \). Leibniz obtained this series in 1674.

Note: \( \tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \)

Euler’s celebrated sum- You can obtain this by finding the Fourier series for \( f(t) = t^2, \ -\pi < t < +\pi, \ f(t+2\pi) = f(t) \).

\[
\begin{align*}
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots &= \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \\
1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{1}{2} \cdot \frac{\pi^2}{6} \\
1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{64} + \cdots &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \\
\sum_{k=1}^{\infty} \frac{1}{k^4} &= \frac{\pi^4}{90}
\end{align*}
\]

No one has come up with the exact value of \( \sum_{k=1}^{\infty} \frac{1}{k^N}, \ N = \text{Odd integer greater than 1.} \)