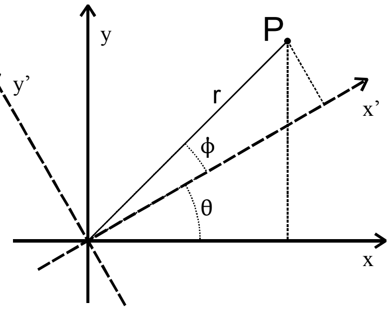
Rotation of Axes



The point P is the same in both the xy and $x'y'$ coordinate systems.

The point can be represented in the xy plane as $x = r\cos(\phi+\theta)$ and $y = r\sin(\phi+\theta)$
The point can be represented in the $x'y'$ plane as $x' = r\cos\phi$ and $y' = r\sin\phi$

Using the sum of angles identities from trigonometry gives us ...

$$\begin{aligned} x &= r\cos(\phi+\theta) \\ &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x'\cos\theta - y'\sin\theta \end{aligned} \qquad \begin{aligned} y &= r\sin(\phi+\theta) \\ &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= x'\sin\theta + y'\cos\theta \end{aligned}$$

Substituting x and y into the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ gives:

$$\begin{aligned} &A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + \\ &C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0 \end{aligned}$$

Expanding the factors gives:

$$\begin{aligned} &A\left(x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta\right) + \\ &B\left(x'^2\cos\theta\sin\theta + x'y'\cos^2\theta - x'y'\sin^2\theta - y'^2\cos\theta\sin\theta\right) + \\ &C\left(x'^2\sin^2\theta + 2x'y'\cos\theta\sin\theta + y'^2\cos^2\theta\right) + \\ &D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0 \end{aligned}$$

Collecting the terms yields:

$$\begin{aligned} &x'^2\left(A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta\right) + \\ &x'y'\left(B\left(\cos^2\theta - \sin^2\theta\right) - 2(A-C)\cos\theta\sin\theta\right) + \\ &y'^2\left(A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta\right) + \\ &x'(D\cos\theta + E\sin\theta) + y'(-D\sin\theta + E\cos\theta) + F = 0 \end{aligned}$$

The double angle identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$ allow us to rewrite the $x'y'$ term.

$$\begin{aligned}
&x'^2\left(A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta\right) + \\
&x'y'\left(B\cos 2\theta - \left(A - C\right)\sin 2\theta\right) + \\
&y'^2\left(A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta\right) + \\
&x'\left(D\cos\theta + E\sin\theta\right) + y'\left(-D\sin\theta + E\cos\theta\right) + F = 0
\end{aligned}$$

The general form can be written as $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ where

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
B' &= B\cos 2\theta - \left(A - C\right)\sin 2\theta \\
C' &= A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= -D\sin\theta + E\cos\theta \\
F' &= F
\end{aligned}$$

The goal is to find the angle of rotation θ such that $B' = 0$.

$$\begin{aligned}
&B\cos 2\theta - \left(A - C\right)\sin 2\theta = 0 \\
&B\cos 2\theta = \left(A - C\right)\sin 2\theta \\
&\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B} \\
&\cot 2\theta = \frac{A - C}{B} \\
&2\theta = \cot^{-1}\left(\frac{A - C}{B}\right) \\
&\theta = \frac{1}{2}\cot^{-1}\left(\frac{A - C}{B}\right)
\end{aligned}$$

Some people will find it easier to use

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{B}{A - C}\right)$$

We don't because it is possible that $A = C$ and you might get division by zero.

We can avoid this by using the inverse cotangent function. We know that B will never be zero since the whole point is to eliminate the xy term.
Also, the range of the inverse cotangent is between $0°$ and $180°$, so $0° < 2\theta < 180°$, which means that $0° < \theta < 90°$.

While it is possible to get an angle using this formula, it is usually easier to draw a triangle with 2θ as the angle and use the half-angle identities to find the values for sin θ and cos θ.

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \text{ and } \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

The discriminant, $B^2 - 4AC$, can be used to determine the type of conic section. The discriminant is not affected by the rotation, so you can use the original coefficients or the new ones after the rotation. The discriminant does not detect the degenerate cases.

If $B^2 - 4AC < 0$, the conic is a circle or an ellipse.
If $B^2 - 4AC = 0$, the conic is a parabola.
If $B^2 - 4AC > 0$, the conic is a hyperbola.

Example Problem

$$2\sqrt{7}x^2 - 2\sqrt{70}xy + 5\sqrt{7}y^2 - 28\left(\sqrt{2} + 2\sqrt{5}\right)x + 28\left(\sqrt{5} - 2\sqrt{2}\right)y - 28\sqrt{7} = 0$$

The discriminant is $B^2 - 4AC = \left(-2\sqrt{70}\right)^2 - 4\left(2\sqrt{7}\right)\left(5\sqrt{7}\right) = 280 - 280 = 0$

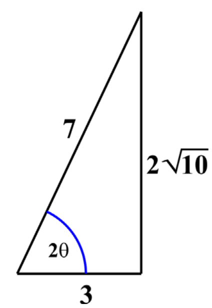
Since the discriminant is 0, this is going to be a parabola.

$$\cot 2\theta = \frac{A - C}{B} = \frac{2\sqrt{7} - 5\sqrt{7}}{-2\sqrt{70}} = \frac{-3\sqrt{7}}{-2\sqrt{70}} = \frac{3}{2\sqrt{10}}$$

If you need to know the angle, then $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2\sqrt{10}}{3} \approx 32.31153324°$

But you don't need to know the angle to get rid of the xy term.

Draw a triangle so that $\cot 2\theta = \dfrac{3}{2\sqrt{10}}$ and use it to determine the values for the cosine and sine using the half angle identities. Use the positive values for $\cos\theta$ and $\sin\theta$ since θ is an acute angle.



$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} = \frac{1 + 3/7}{2} = \frac{5}{7} \qquad \cos\theta = \sqrt{\frac{5}{7}}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} = \frac{1 - 3/7}{2} = \frac{2}{7} \qquad \sin\theta = \sqrt{\frac{2}{7}}$$

Make the substitutions into the formulas above.

$$A' = 2\sqrt{7}\left(\frac{5}{7}\right) + \left(-2\sqrt{70}\right)\left(\sqrt{\frac{5}{7}}\right)\left(\sqrt{\frac{2}{7}}\right) + 5\sqrt{7}\left(\frac{2}{7}\right) = \frac{10}{\sqrt{7}} - \frac{20}{\sqrt{7}} + \frac{10}{\sqrt{7}} = 0$$

$$B' = -2\sqrt{70}\left(\frac{3}{7}\right) - \left(2\sqrt{7} - 5\sqrt{7}\right)\left(\frac{2\sqrt{10}}{7}\right) = -\frac{6\sqrt{10}}{\sqrt{7}} + \frac{6\sqrt{10}}{\sqrt{7}} = 0$$

$$C' = 2\sqrt{7}\left(\frac{2}{7}\right) - \left(-2\sqrt{70}\right)\left(\sqrt{\frac{5}{7}}\right)\left(\sqrt{\frac{2}{7}}\right) + 5\sqrt{7}\left(\frac{5}{7}\right) = \frac{4}{\sqrt{7}} + \frac{20}{\sqrt{7}} + \frac{25}{\sqrt{7}} = \frac{49}{\sqrt{7}}$$

$$D' = -28\left(\sqrt{2} + 2\sqrt{5}\right)\left(\sqrt{\frac{5}{7}}\right) + 28\left(\sqrt{5} - 2\sqrt{2}\right)\left(\sqrt{\frac{2}{7}}\right)$$

$$= -\frac{28}{\sqrt{7}}\left(\sqrt{10} + 10 - \sqrt{10} + 4\right) = -\frac{392}{\sqrt{7}}$$

$$E' = -\left(-28\left(\sqrt{2} + 2\sqrt{5}\right)\right)\left(\sqrt{\frac{2}{7}}\right) + 28\left(\sqrt{5} - 2\sqrt{2}\right)\left(\sqrt{\frac{5}{7}}\right)$$

$$= \frac{28}{\sqrt{7}}\left(2 + 2\sqrt{10} + 5 - 2\sqrt{10}\right) = \frac{196}{\sqrt{7}}$$

$$F' = -28\sqrt{7}$$

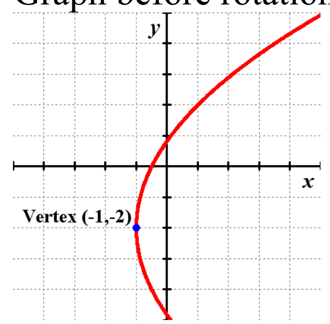
The new equation in the $x'y'$ plane is $\frac{49}{\sqrt{7}}y'^2 - \frac{392}{\sqrt{7}}x' + \frac{196}{\sqrt{7}}y' - 28\sqrt{7} = 0$.

Multiplying everything by $\frac{\sqrt{7}}{49}$ gives $y'^2 - 8x' + 4y' - 4 = 0$.

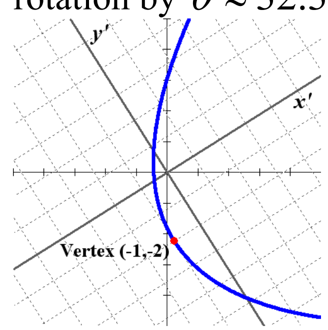
Notice that the discriminant is still zero.

$$B'^2 - 4A'C' = 0^2 - 4(0)(1) = 0$$

Graph before rotation.



Same graph after rotation by $\theta \approx 32.3°$.



If you put general equation into standard form by completing the square, you get

$$y'^2 + 4y = 8x' + 4$$
$$y'^2 + 4y + 4 = 8x' + 4 + 4$$
$$(y' + 2)^2 = 8(x' + 1)$$
$$(y' + 2)^2 = 4(2)(x' + 1)$$

This is a parabola with a vertex at $(-1,-2)$, opening along the positive x' axis, with a focal length of 2. That puts the focus at $(1,-2)$ and the directrix at $x' = -3$.

Use $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{2}/\sqrt{7}}{\sqrt{5}/\sqrt{7}} = \frac{\sqrt{2}}{\sqrt{5}}$ to get the correct angle on the rotated axes.