Simplify the radical expressions

1. $$\frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

2. $$\sqrt{\frac{8}{5}} = \frac{2\sqrt{2}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{10}}{5}$$

3. $$\sqrt[3]{\frac{5}{2}} = \sqrt[3]{\frac{5}{2}}\sqrt[3]{\frac{4}{4}} = \frac{\sqrt[3]{20}}{2}$$

4. $$\sqrt{4x^2y} = 2x\sqrt{y}$$

5. $$\sqrt[3]{81x^4y^5} = 3xy\sqrt[3]{3xy^2}$$

6. $$\frac{3-\sqrt{2}}{1+3\sqrt{5}} = \left(\frac{3-\sqrt{2}}{1+3\sqrt{5}}\right)\left(\frac{1-3\sqrt{5}}{1-3\sqrt{5}}\right)$$
$$= \frac{3-9\sqrt{5}-\sqrt{2}+3\sqrt{10}}{1-45} = -\frac{3-9\sqrt{5}-\sqrt{2}+3\sqrt{10}}{44}$$

7. $$\frac{x-3}{x-\sqrt{3}} = \left(\frac{x-3}{x-\sqrt{3}}\right)\left(\frac{x+\sqrt{3}}{x+\sqrt{3}}\right) = \frac{x^2-3x+x\sqrt{3}-3\sqrt{3}}{x^2-3}$$

Expand and simplify

8. $$(3x-2)(5x+7) = 15x^2+11x-14$$

9. $$\left(1-x^{-1}\right)(2+5x) = 2+5x-2x^{-1}+5 = 5x+7-2x^{-1}$$

10. $$1+\left(x^3-x^{-3}\right)^2 = 1+x^6-2+x^{-6} = x^6-1+x^{-6}$$

11. $$(3x-2)^3 = 1(3x)^3(-2)^0+3(3x)^2(-2)^1+3(3x)^1(-2)^2+1(3x)^0(-2)^3$$
$$= 27x^3-54x^2+36x-8$$

Simplify the expressions

12. $\dfrac{8x^3+5x^2-3x^{-1}}{4x}=2x^2+\dfrac{5}{4}x-\dfrac{3}{4}x^{-2}$

13. $\dfrac{x^2-5x+6}{x-3}=\dfrac{(x-3)(x-2)}{x-3}=x-2$

14. $\dfrac{5(x-2)^2-4x(x-2)}{(x-2)^3}=\dfrac{5(x-2)-4x}{(x-2)^2}=\dfrac{x-10}{(x-2)^2}$

15. $\left(3x^2y^{-1}\right)^4=81x^8y^{-4}$

16. $\left(2x^{-3}y^2\right)^{-2}\left(5xy^{-3}\right)^3=\left(\dfrac{1}{4}x^6y^{-4}\right)\left(125x^3y^{-9}\right)=\dfrac{125x^9}{4y^{13}}$

17. $\dfrac{8x^{1/3}y^{9/5}}{3x^{4/3}y^{2/5}}=\dfrac{8y^{\frac{9}{5}-\frac{2}{5}}}{3x^{\frac{4}{3}-\frac{1}{3}}}=\dfrac{8y^{7/5}}{3x}$

18. $\dfrac{4t^{9/4}s^{3/5}}{5t^{1/3}s^{-1/4}}=\dfrac{4}{5}t^{\frac{9}{4}-\frac{1}{3}}s^{\frac{3}{5}+\frac{1}{4}}=\dfrac{4}{5}t^{23/12}s^{17/20}$

Evaluate and simplify

19. $\dfrac{5\pm\sqrt{(-5)^2-4(3)(-2)}}{2(3)}=\dfrac{5\pm\sqrt{49}}{6}=\dfrac{5\pm7}{6}=\dfrac{12}{6},-\dfrac{2}{6}=2,-\dfrac{1}{3}$

20. $2x^2(3x-1)\sqrt{5x-1}\Big|_{x=2}=2(2)^2(3(2)-1)\sqrt{5(2)-1}=2(4)(5)\sqrt{9}=120$

21. $4x^{3/4}(x-1)^2\Big|_{x=16}=4(16)^{3/4}(16-1)^2=4(8)(15)^2=7200$

Factor completely

22. $10x^3-13x^2-3x=x\left(10x^2-13x-3\right)=x(2x-3)(5x+1)$

23. $10x^3-13x^2+3x=x\left(10x^2-13x+3\right)=x(x-1)(10x-3)$

24. $4x^{7/3}-2x^{4/3}-12x^{1/3}=2x^{1/3}\left(2x^2-x-6\right)=2x^{1/3}(x-2)(2x+3)$

25. $3x^4-5x^3-12x^2+20x=x\left(3x^3-5x^2-12x+20\right)$

$$=x\left[x^2(3x-5)-4(3x-5)\right]=x\left(x^2-4\right)(3x-5)$$

$$=x(x-2)(x+2)(3x-5)$$

26. $8x^3-27=(2x-3)\left((2x)^2(3)^0+(2x)^1(3)^1+(2x)^0(3)^2\right)$

$$=(2x-3)\left(4x^2+6x+9\right)$$

27. $2x^3+7x^2-14x+5=(x-1)\left(2x^2+9x-5\right)=(x-1)(2x-1)(x+5)$

Hint: Find one root and then use synthetic division.

Since the coefficients add to zero, we know that $x=1$ *is a root.*

$$\begin{array}{r|rrrr} 1 & 2 & 7 & -14 & 5 \\ & & 2 & 9 & -5 \\ \hline & 2 & 9 & -5 & \underline{|0} \end{array}$$

28. $x^4-3x^3-4x^2-3x-5=\left(x^2-3x-5\right)\left(x^2+1\right)$

Hint: graph, approximate the roots numerically, and work backwards to find two of the roots. Then use polynomial division to find the other factor(s).

- Using a graphing utility, we find that two of the roots are -1.192582 and 4.192582.
- These are not rational roots and when the coefficients are real, the irrational roots come in pairs, so if $a+\sqrt{b}$ is a root, so is $a-\sqrt{b}$.
- $$\begin{cases} a + \sqrt{b} = 4.192582 \\ a - \sqrt{b} = -1.192582 \end{cases}$$
- If you add the two equations together, you get $2a=3 \Rightarrow a=\dfrac{3}{2}$
- If you subtract the two equations, you get $2\sqrt{b}=5.385164 \Rightarrow b=7.25=\dfrac{29}{4}$.
- Thus $x=\dfrac{3\pm\sqrt{29}}{2} \Rightarrow 2x=3\pm\sqrt{29} \Rightarrow 2x-3=\pm\sqrt{29}$

$$\Rightarrow (2x-3)^2=\left(\pm\sqrt{29}\right)^2 \Rightarrow 4x^2-12x+9=29 \Rightarrow 4x^2-12x-20=0$$

$$\Rightarrow x^2-3x-5=0$$

- So one of the factors is x^2-3x-5 and you can use polynomial division to find the other one.

$$\begin{array}{r} x^2 \quad + \quad 1 \\ x^2-3x-5\overline{\smash{)}x^4-3x^3-4x^2-3x-5} \\ \underline{x^4-3x^3-5x^2} \qquad\qquad \\ x^2-3x-5 \end{array}$$

Use the table to evaluate

x	-2	-1	0	1	2
$f(x)$	3	2	-1	2	0
$g(x)$	2	1	5	-2	-1

29. $(f+g)(-2) = f(-2)+g(-2) = 3+2 = 5$

30. $(fg)(0) = f(0)\times g(0) = -1(5) = -5$

31. $f\left[g(1)\right] = f(-2) = 3$

32. $(g\circ f)(1) = g\left[f(1)\right] = g(2) = -1$

33. $f^2(5x-2)\Big|_{x=0} = \left[f(5\cdot 0-2)\right]^2 = \left[f(-2)\right]^2 = (3)^2 = 9$

34. $x^2 f(3x+2)g(1-x)\Big|_{x=-1} = (-1)^2\, f\left[3(-1)+2\right]g\left[1-(-1)\right]$

$$= 1f(-1)g(2) = 2(-1) = -2$$

Find all real solutions

35. $(x-2)^2(3x-1)^3(5x^2-9)(x^2+4) = 0 \Rightarrow \left\{2, \frac{1}{3}, \pm\frac{3}{\sqrt{5}}\right\}$

$x-2=0 \text{ or } 3x-1=0 \text{ or } 5x^2-9=0 \text{ or } x^2+4=0$

$x=2,\ x=\frac{1}{3},\ x^2=\frac{9}{5},\ x^2=-4$

The square roots of -4 are complex, not real.

36. $(x+3)^{1/2}(3x+1)^{-2}(x+7)^3(5x-3)^{1/3} = 0 \Rightarrow \left\{-3, \frac{3}{5}\right\}$

- The $(x+3)^{1/2} = \sqrt{x+3}$ so this gives us $x=-3$ but also restricts the domain to be $x \ge -3$.
- The $(3x+1)^{-2}$ gives us a vertical asymptote at $x=-\frac{1}{3}$ because the factor is in the denominator.
- The $(x+7)^3$ gives us $x=-7$, but that doesn't fall in the domain of $x \ge -3$
- The $(5x-3)^{1/3}$ gives us $x=\frac{3}{5}$

Piecewise Functions

Evaluate each of the following when $f(x)=\begin{cases}3x-5, & x<3\\ x^2-9 & x>3\end{cases}$

37. $f(2)=3(2)-5=1$

38. $f(3)$ is undefined

39. $f(4)=4^2-9=7$

Simplify by factoring

40. $(3x+2)^2(5x-1)^5+(3x+2)^3(5x-1)^4$

$$=(3x+2)^2(5x-1)^4\left[(5x-1)+(3x+2)\right]$$

$$=(3x+2)^2(5x-1)^4(8x+1)$$

41. $24x\left(3x^2-1\right)^3(4x+3)^3+12\left(3x^2-1\right)^4(4x+3)^2$

$$=12\left(3x^2-1\right)^3(4x+3)^2\left[2x(4x+3)+\left(3x^2-1\right)\right]$$

$$=12\left(3x^2-1\right)^3(4x+3)^2\left(11x^2+6x-1\right)$$

42. $\tfrac{32}{3}(3x-1)^{5/4}(4x+3)^{5/3}+\tfrac{15}{4}(3x-1)^{1/4}(4x+3)^{8/3}$

$$=\frac{1}{12}(3x-1)^{1/4}(4x+3)^{5/3}\left[128(3x-1)+45(4x+3)\right]$$

$$=\frac{1}{12}(3x-1)^{1/4}(4x+3)^{5/3}(564x+7)$$

43. $\tfrac{9}{4}(2x+1)^{1/3}(3x+2)^{-1/4}+\tfrac{2}{3}(2x+1)^{-2/3}(3x+2)^{3/4}$

$$=\frac{1}{12}(2x+1)^{-2/3}(3x+2)^{-1/4}\left[27(2x+1)+8(3x+2)\right]$$

$$=\frac{1}{12}(2x+1)^{-2/3}(3x+2)^{-1/4}(78x+43)$$

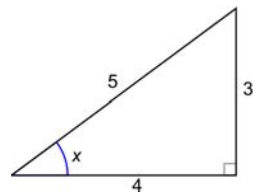
44. $\dfrac{10(3x-4)^3(5x+3)-9(5x+3)^2(3x-4)^2}{(3x-4)^6}$

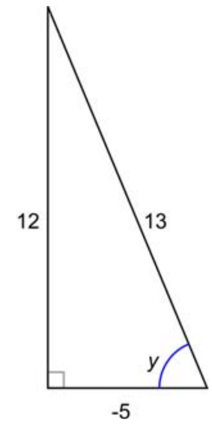
$$=\frac{(5x+3)}{(3x-4)^4}\left[10(3x-4)-9(5x+3)\right]$$

$$=\frac{(5x+3)(-15x-67)}{(3x-4)^4}=-\frac{(5x+3)(15x+67)}{(3x-4)^4}$$

Trigonometric Identities

There are two angles, *x* and *y*, such that $\sin x = 3/5$, $\tan y = -12/5$, $0 < x < \pi/2$, and $\pi/2 < y < \pi$. Draw two appropriate triangles and use them to find the following.





45. $\cos x = \frac{4}{5}$

46. $\sin(\pi - x) = \sin x = \frac{3}{5}$

47. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{\frac{3}{4} - \left(-\frac{12}{5}\right)}{1 + \frac{3}{4}\left(-\frac{12}{5}\right)} = \frac{15+48}{20-36} = \frac{63}{-16} = -\frac{63}{16}$

48. $\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2} = \frac{1+\frac{4}{5}}{2} = \frac{9}{10}$

49. $\sin 2y = 2\sin y \cos y = 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right) = -\frac{120}{169}$

50. $\cos(y-x) = \cos y \cos x + \sin y \sin x = -\frac{5}{13}\left(\frac{4}{5}\right) + \frac{12}{13}\left(\frac{3}{5}\right) = \frac{16}{65}$

51. $\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(\frac{12}{13}\right) = \frac{33}{65}$

52. $\cos^2(5x) + \sin^2(5x) = 1$

53. $\sin x \sec y = \frac{3}{5}\left(-\frac{13}{5}\right) = -\frac{39}{25}$

54. $\sec(\pi + x) = -\sec x = -\frac{5}{4}$

55. $\cot(-y)\ \cot(-y) = -\cot(y) = -\left(-\frac{12}{5}\right) = \frac{12}{5}$

56. $\sec^2 y + \csc^2 x = \left(-\frac{13}{5}\right)^2 + \left(\frac{5}{3}\right)^2 = \frac{169}{25} + \frac{25}{9} = \frac{2146}{225}$

Manipulate and Evaluate

All of these problems are of the form $0/0$ when you try to evaluate them using direct substitution at the indicated point. Algebraically manipulate the expression to eliminate the 0 from the denominator and then evaluate it at the indicated point.

57. $\dfrac{5x}{x^2+2x}, \quad x=0$

$$\frac{5x}{x(x+2)}=\frac{5}{x+2}\Rightarrow\frac{5}{0+2}=\frac{5}{2}$$

58. $\dfrac{5x^2-13x-6}{x-3}, \quad x=3$

$$\frac{(x-3)(5x+2)}{x-3}=5x+2\Rightarrow 5(3)+2=17$$

59. $\dfrac{(x+h)^5-x^5}{h}, \quad h=0$

$$\frac{(x+h-x)\left[(x+h)^4+(x+h)^3x+(x+h)^2x^2+(x+h)x^3+x^4\right]}{h}$$

$$=(x+h)^4+(x+h)^3x+(x+h)^2x^2+(x+h)x^3+x^4$$

$$\Rightarrow x^4+x^3x+x^2x^2+xx^3+x^4=5x^4$$

Here is another way to work the problem. Instead of factoring, you can expand.

$$\frac{(x+h)^5-x^5}{h}=\frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h}$$

$$=\frac{5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5}{h}$$

$$=5x^4+10x^3h+10x^2h^2+5xh^3+h^4\Rightarrow 5x^4$$

60. $\dfrac{\sqrt{x+4}-3}{x-5},\ \ x=5$

$$\left(\frac{\sqrt{x+4}-3}{x-5}\right)\left(\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}\right)=\frac{x+4-9}{(x-5)\left(\sqrt{x+4}+3\right)}=\frac{1}{\sqrt{x+4}+3}$$

$$\Rightarrow \frac{1}{\sqrt{5+4}+3}=\frac{1}{6}$$

61. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h},\ \ h=0$

$$\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)=\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\Rightarrow \frac{1}{2\sqrt{x}}$$

62. $\dfrac{\frac{3}{x}-\frac{3}{4}}{x-4},\ \ x=4$

$$\left(\frac{\frac{3}{x}-\frac{3}{4}}{x-4}\right)\left(\frac{4x}{4x}\right)=\frac{3(4-x)}{4x(x-4)}=-\frac{3}{4x}\Rightarrow -\frac{3}{4(4)}=-\frac{3}{16}$$

63. $\dfrac{1-\cos x}{\sin x},\ \ x=0$

$$\left(\frac{1-\cos x}{\sin x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)=\frac{1-\cos^2 x}{\sin x(1+\cos x)}=\frac{\sin^2 x}{\sin x(1+\cos x)}=\frac{\sin x}{1+\cos x}$$

$$\Rightarrow \frac{\sin 0}{1+\cos 0}=\frac{0}{1+1}=0$$