

# Math 160 - Mathematical Notation

## Purpose

One goal in any course is to properly use the language of that subject. Finite Mathematics is no different but may often seem like a foreign language. These notations summarize some of the major concepts or more difficult topics of the unit. Typing them helps you learn the material while teaching you to properly express mathematics on the computer. Part of your grade is for properly using mathematical content.

## Instructions

Use Microsoft Word to recreate the following documents and submit them through the Canvas learning management system. Save the Word documents as .docx format; do not submit .rtf or .pdf files.

There will be one document for each chapter. The due dates are listed in Canvas, but you may turn them in early if you like. If you're going to turn in the notations early, it is recommended that you submit the first one and get feedback before you complete all of them. Late work will be accepted but will receive a 20% per day penalty. No late work will be accepted after the final.

Place your name in the heading of the document so that it appears on each page. Failure to put your name in the document will cost you points. Include the chapter and section titles as part of what you type.

You do not have to match the formatting (line breaks, font sizes, colors, etc.) that is used in this document. The instructor did not use Word and you should not expect it to match exactly.

Non-mathematical content should be typed as normal. When you come to mathematical content, you should use the equation editor in Word by selecting the Insert toolbar from the menu and clicking on Equation (the  $\pi$  symbol) or using the keyboard shortcut  $\text{Alt}=\text{}$ .

The equation editor in Microsoft Word changed with Office 2007 and is friendlier for the masses but not as powerful as the older versions. The instructor is not using Word to create these documents, which means that you will be unable to recreate some of the content here exactly.

If there is an equation, put both sides of the equation into the same equation editor box instead of creating two objects. Be sure to use the proper symbols, there are some instances where more than one symbol may look the same, but they have different meanings and don't appear the same as what's on the assignment.

Use balanced parentheses from the toolbar so fractions look like  $\left(\frac{1}{3}\right)$  instead of  $(\frac{1}{3})$ .

There are a lot of matrices and systems of equations in this course. Matrices, especially those with partition lines in them, are challenging with the built-in equation editor via Insert  $\rightarrow$  Equation. However, it is the only option available without learning a completely new system, so insert the vertical line and ignore the horizontal one.

Many of the systems of equations have been placed inside of a matrix container for spacing purposes.

There are additional tips on the instructor's website at <https://people.richland.edu/james/editor/>.

## Grading

Points will be awarded according to the following rubric.

Criteria	Pts	Description
Mathematics	4	Used equation objects correctly
	3	Used correctly with a few minor issues
	2	On right track, but has some major issues
	1	Very little use of objects
	0	Didn't use mathematical objects
Completeness	4	Entire document is present
	3	Missing an occasional word
	2	Missing sentences
	1	Missing sections
	0	Missing majority of document
Formatting	2	Name and title both present
	1	Missing name or title
	0	Missing both name and title

## Notes

**This is not a group assignment.** Each person needs to create and submit their own notation. If it is determined that sharing occurs, a reduced grade will result for all parties involved.

Every now and then you will find formatting tips in boxes like this one.  
Do not type these as part of your document.

The point is to construct the document from scratch so that you learn the material and how to type mathematical content. I need to be able to verify that you're using the equation editor. I should be able to open the document in Word and edit your equation.

Do not import the PDF file into Word, change a few things, and pass it off as your work. The transition is not as smooth as you would hope. Tips that you've done this include equations that are converted to images, including the yellow comment boxes (do not do this), and not using the equation editor. Doing this will get you no credit for the assignment.

Look at the rubric and follow the instructions. Many students have lost points for something simple like leaving their name off the paper.

All work must be original and created this semester. Work submitted in a previous semester or for a different course will not be accepted.

## Chapter 4 - Systems of Equations, Matrices

### Row Operations

These are operations that produce row equivalent matrices:

1. Switch two rows of a matrix:  $R_1 \leftrightarrow R_2$
2. Multiply a row by a non-zero constant:  $3R_1 \rightarrow R_1$
3. Add a constant multiple of one row to another row:  $3R_1 + R_2 \rightarrow R_2$

### Row-Echelon Form (REF)

A matrix is in row-echelon form when:

1. If there is a row of all zeros, then it is at the bottom.
2. The first element in any row that is not a zero is a one. This is called the leading one.
3. All elements below a leading one are zero. The leading one of any row must be to the right of the leading one of the previous row.

### Reduced Row-Echelon Form (RREF)

A matrix is in reduced row-echelon form when all of the conditions of row-echelon form are met and:

4. The column with a leading one is cleared and all elements above and below the leading one are zero.

### Matrix Multiplication

Multiply the rows from the first matrix by the columns from the second matrix and add their products together.

The projects page in Canvas has hints and videos on how to enter matrices.

$$\begin{bmatrix} 1 & -3 & 4 \\ -7 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 5 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 42 & -26 \\ 13 & -12 \end{bmatrix}$$

To align the columns, place the following inside a table without borders.

Do not type these hints.

$$\begin{array}{l} R_1 \times C_1 : \quad 1(2) - 3(-4) + 4(7) \quad = \quad 2 + 12 + 28 \quad = \quad 42 \\ R_1 \times C_2 : \quad 1(1) - 3(5) + 4(-3) \quad = \quad 1 - 15 - 12 \quad = \quad -26 \\ R_2 \times C_1 : \quad -7(2) + 2(-4) + 5(7) \quad = \quad -14 - 8 + 35 \quad = \quad 13 \\ R_2 \times C_2 : \quad -7(1) + 2(5) + 5(-3) \quad = \quad -7 + 10 - 15 \quad = \quad -12 \end{array}$$

### Leontief Input-Output Model

$$\mathbf{X} = (\mathbf{I} - \mathbf{M})^{-1} \mathbf{D}$$

$\mathbf{X}$  = Output Matrix,  $\mathbf{M}$  = Technology Matrix,  $\mathbf{D}$  = Demand Matrix

# Chapter 5 - Linear Inequalities

## System of Linear Inequalities

Use a matrix and leave some cells empty to get the proper alignment.

$$\begin{array}{rcl} 2x + 5y & \leq & 30 \\ 4x + 3y & \leq & 20 \\ x & \geq & 0 \\ & y & \geq 0 \end{array}$$

## Lines and Shading

The boundary lines of the region are obtained by using the equal sign.

Use a test point not on the line and plug it into the inequality to determine shading.

When all the coefficients are positive, then  $>$  or  $\geq$  inequalities will be shaded above or to the right of the line and  $<$  or  $\leq$  inequalities will be shaded below or to the left of the line.

## Definitions

- The **feasible region** is the region where all of the constraints are satisfied.
- A **corner point** is a vertex of the feasible region.
- A **bounded** feasible region can be enclosed in a circle.
- An **unbounded** feasible region cannot be enclosed in a circle.

## Existence of Solutions to a Linear Programming Problem

- For a bounded feasible region, the objective function will always have both a maximum and minimum.
- For an unbounded feasible region with positive coefficients of the objective function, there will be a minimum but no maximum value.
- If the feasible region is empty, then the objective function has no maximum or minimum value.

## Fundamental Theorem of Linear Programming

If there is a solution to a linear programming problem, then it will occur at one or more corner points of the feasible region or on the boundary between two corner points.

## Non-Negativity Constraints

Although usually not written in the problem itself, almost every story problem has **non-negativity constraints**. These state that the variables cannot be negative and are written as  $x \geq 0$  and  $y \geq 0$ . These non-negativity constraints limit the feasible region to the first quadrant.

## Chapter 6 - Linear Programming

### Standard Problems

A **standard maximization problem** is a maximization problem where all problem constraints are  $\leq$  a non-negative constant and the objective function coefficients can be any real number.

A **standard minimization problem** is a minimization problem where all problem constraints are  $\geq$  any real number, but the objective function coefficients cannot be negative. The technique for solving a standard minimization problem is to form the dual problem, which is a standard maximization problem, use the Simplex technique, and read the solution from the bottom row.

### Standard Maximization Problem

$$\text{Maximize } P = 300x_1 + 200x_2 + 400x_3$$

Subject to

$$\begin{array}{rclclcl} x_1 & + & x_2 & + & x_3 & \leq & 200 \\ 5x_1 & + & 2x_2 & + & 4x_3 & \leq & 300 \\ x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

### Standard Maximization Initial System

The initial system is what you get after adding slack variables.

$$\text{Maximize } P = 300x_1 + 200x_2 + 400x_3$$

Subject to

$$\begin{array}{rclclclcl} x_1 & + & x_2 & + & x_3 & + & s_1 & = & 200 \\ 5x_1 & + & 2x_2 & + & 4x_3 & & & + & s_2 & = & 300 \\ x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & \geq & 0 \end{array}$$

### Standard Maximization Initial Tableau

Move the objective function variables to the left-hand side and put it into matrix form.

Word's equation editor will not let you put the **horizontal** augmentation line in there. Don't stress over it.

$$\left[ \begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 200 \\ 5 & 2 & 4 & 0 & 1 & 0 & 300 \\ \hline -300 & -200 & -400 & 0 & 0 & 1 & 0 \end{array} \right]$$

### Non-Standard Problems

For non-standard problems, do the following

- For minimization problems, multiply the objective function by  $-1$  and then maximize.
- For each  $\leq$  constraint, add a slack variable to the constraint.
- For each  $\geq$  constraint, subtract a surplus variable from the constraint, add an artificial variable to the constraint, and subtract  $M$  times the artificial variable from the objective function.
- For each  $=$  constraint, add an artificial variable to the constraint and subtract  $M$  times the artificial variable from the objective function.

# Chapter 7 - Logic, Sets, and Counting

## Logic

- A logical **and**  $p \wedge q$  is true only when both statements are true and false if any part is false.
- A logical **or**  $p \vee q$  is false only when both statements are false and true if any part is true.
- A logical **implies**  $p \rightarrow q$  is false only when  $p$  is true and  $q$  is false.

## Sets

- The **union**  $A \cup B$  of two sets contains anything found in either set.
- The **intersection**  $A \cap B$  of two sets contain only those items found in both sets.

## Counting

### Fundamental Counting Principle

The total number of ways that two events can happen is found by multiplying together the number of ways that each event can happen.

### Factorial

For positive integers, a factorial represents the number of ways you can arrange that many objects without repetition but with regard to order. Written using an exclamation point,  $n! = n(n-1)(n-2) \cdots (3)(2)(1)$

For example,  $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ .

$0! = 1$  is a special case. There is only 1 way to select 0 items.

### Permutation

A **permutation** can be written as  $nPr$  or  $P(n, r)$ . It represents the number of unique ways you can arrange  $r$  items taken from a group of  $n$  items. There is no repetition allowed and the order of the items is important.

$$P(n, r) = \frac{n!}{(n-r)!}$$

### Combination

See the equation editor handout and videos for tips on how to enter the combination notations.

A **combination** can be written as  $\binom{n}{r}$ ,  ${}_nC_r$ , or  $C(n, r)$ . It represents the number of unique ways that you can take a group of  $n$  items and separate them into two groups of size  $r$  and  $n-r$ . There is no repetition allowed, but the order of the items is not important.

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

### Partitioning

**Partitioning** allows you to extend a combination beyond two groups. The important part is that you account for every item, which means that values on the bottom add to the value on the top.

For example, the number of ways you can divide 15 people into teams of 5, 4, 3, and 3 people each is

$$\frac{15!}{5! \times 4! \times 3! \times 3!} = 12,612,600$$

# Chapter 8 - Probability

## Probability Rules

The following assume the universal set is  $U$ .

1. All probabilities are between 0 and 1, inclusive.  $0 \leq P(A) \leq 1$
2. The sum of the probabilities of all the different outcomes is 1.  $\sum P(A) = 1$
3. The probability of something that is impossible is 0.  $P(\emptyset) = 0$
4. The probability of something that is certain is 1.  $P(U) = 1$
5. The probability of something **not** happening is one minus the probability of it happening.  
 $P(A') = 1 - P(A)$
6. The probability of **any** one of several disjoint items happening is the sum of their probabilities.  
 $P(A \cup B) = P(A) + P(B)$   
If the items are not disjoint, then you need to subtract their overlap.  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
7. The probability of **all** of several independent items happening is the product of their probabilities.  
 $P(A \cap B) = P(A)P(B)$ . If the events are not independent then you need to use conditional probabilities.  
 $P(A \cap B) = P(A)P(B|A)$

## Expected Value

The expected value of a probability distribution is also called its mean or average. It is found by multiplying the values by their associated probabilities and then adding those products together.

$$E(x) = \sum xp$$

## Decision Theory

### Expected value (Bayesian) criterion

This is the only criterion covered that uses probabilities. It is useful when repeating a process many times. It is the criterion that would be chosen by a *realist*. You find the expected value under each action and choose the action with the largest expected value.

### Maximax criterion

This is the best of the best case scenarios, the criterion an *optimist* would choose. You find the maximum payoff under each action and then choose the action with the largest best case scenario.

### Maximin criterion

This is the best of the worst case scenarios, the criterion a *pessimist* would choose. Find the minimum payoff under each action and then choose the action with the largest worst case scenario.

### Minimax criterion

This method finds the regret or opportunistic loss for each state of nature. These aren't real losses, they are perceived losses. This method uses losses, so smaller is better. The *opportunist* chooses the best of the worst losses for each action. You first find the opportunistic loss for each state of nature and then find the maximum opportunistic loss for each action and choose the action with the smallest (minimum) loss.

## Chapter 9 - Markov Chains

$\mathbf{P}$  is the transition matrix; it is a square matrix and represents the probability of going from the row state to the column state. Both are probability matrices, meaning the sum of each row is 1.

$\mathbf{S}_n$  is a state matrix. It is a single row and the subscript indicates the number of transitions that have occurred.

$\mathbf{S}_0$  is the initial state matrix and it represents the beginning conditions.

$\mathbf{S}_n = \mathbf{S}_0 \mathbf{P}^n$  represents the state after  $n$  transitions.

### Regular Markov Chains

A regular Markov chain is a Markov chain where some power of the transition matrix  $\mathbf{P}$  has all positive entries. It will have a steady-state matrix.

### Steady-State Matrix / Stationary Matrix

The steady-state matrix,  $\mathbf{S}$ , represents the long term probability of being in each state. Once you reach the steady-state matrix, additional transitions do not change the probabilities,  $\mathbf{S} = \mathbf{S}\mathbf{P}$ .

When the Markov chain is regular, then each row of the limiting matrix  $\bar{\mathbf{P}} = \mathbf{P}^\infty$  will be the steady-state matrix.

### Absorbing Markov Chains

Absorbing Markov Chains have some states that transition back to themselves 100% of the time. You must be able to eventually reach some absorbing state from every transient state.

#### Transition Matrix

The standard form for the transition matrix is to list all absorbing states first and all transient states second. This leads to a natural partitioning, where sub-matrices are defined for each section.

Word's equation editor will not let you put the **horizontal** line in the matrix.

$$\mathbf{P} = \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{array} \right]$$

#### Fundamental Matrix

The fundamental matrix  $\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1}$  gives expected frequencies and answers the question "how long?"

It tells you how many times you can expect to spend in the column state, when you start in the row state, before reaching some absorbing state.

#### Long Term Probabilities

The matrix  $\mathbf{FR}$  contains the long term probabilities of ending up in the absorbing column state when you start in the transient row state. It should be used to answer any questions about "eventual probability" or "long term probability".

$\mathbf{FR}$  can also be found from the lower-left block of the limiting matrix  $\bar{\mathbf{P}}$ .

$$\bar{\mathbf{P}} = \mathbf{P}^\infty = \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \mathbf{FR} & \mathbf{0} \end{array} \right]$$

## Chapter 10 - Game Theory

A two-player zero-sum game is composed of two players (row and column) where a win for one player is a loss for the other. No money enters or leaves the game.

The payoff table  $\mathbf{M}$  represents the payoffs for the row player and losses for the column player.

### Strictly Determined Games

A game is strictly determined when any value is both the smallest value in its row and the largest value in its column. That value is the saddle value and is the value of the game.

### $2 \times 2$ Non-Strictly Determined Game

For a  $2 \times 2$  non-strictly determined game where  $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , find  $D = (a + d) - (b + c)$ .

The optimal row player strategy is  $\mathbf{P}^* = \begin{bmatrix} \frac{d-c}{D} & \frac{a-b}{D} \end{bmatrix}$

The optimal column player strategy is  $\mathbf{Q}^* = \begin{bmatrix} \frac{d-b}{D} \\ \frac{a-c}{D} \end{bmatrix}$

The value of the game is  $v = \frac{ad - bc}{D}$

### Linear Programming Problem

Assume that  $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$  has all positive entries. If not, then add a constant  $K$  to all the values first.

Columns are under the Layout tab in Word. You can also put the column player under the row player.

#### Row Player Strategy

The optimal row strategy  $\mathbf{P}^* = [p_1 \ p_2]$  is found by solving this standard minimization problem.

Minimize  $z = \frac{1}{v} = x_1 + x_2$  subject to

$$\begin{array}{rcl} ax_1 + dx_2 & \geq & 1 \\ bx_1 + ex_2 & \geq & 1 \\ cx_1 + fx_2 & \geq & 1 \\ x_1, x_2 & \geq & 0 \end{array}$$

Find  $v = \frac{1}{z}$ , then  $p_1 = x_1v$  and  $p_2 = x_2v$ .

#### Column Player Strategy

The optimal column strategy  $\mathbf{Q}^* = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$  is found

by solving this standard maximization problem.

Maximize  $z = \frac{1}{v} = y_1 + y_2 + y_3$  subject to

$$\begin{array}{rcl} ay_1 + by_2 + cy_3 & \leq & 1 \\ dy_1 + ey_2 + fy_3 & \leq & 1 \\ y_1, y_2, y_3 & \geq & 0 \end{array}$$

Find  $v = \frac{1}{z}$ , then  $q_1 = y_1v$ ,  $q_2 = y_2v$ , and  $q_3 = y_3v$

#### Value of the Game

The value of the original game is  $v = \frac{1}{z} - K$ , where  $K$  is the constant you added at the beginning to make the values all positive.

## Chapter 3 - Finance

Variables representing monetary amounts are in capital letters.

### Simple Interest

Distinguishing feature: a single payment where interest is not compounded. Uses a 360 day year.

$$I = Prt$$

$I$  = Interest,  $P$  = Principal,  $r$  = Rate,  $t$  = Time

Be sure to use the equation editor even for simple equations like  $I = Prt$ . You do not have to use the equation editor for the variable definitions where you are mixing math with text.

### Compound Interest

Distinguishing feature: a single payment where interest is compounded.

$$A = P(1 + i)^n$$

$A$  = Amount,  $P$  = Principal,  $i$  = Periodic Rate,  $n$  = Number of periods

$$r_e = (1 + i)^m - 1$$

$r_e$  = Effective Rate,  $i$  = Periodic Rate,  $m$  = number of periods per year

### Future Value Annuities

Distinguishing feature: a series of payments where the balance grows in value over time.

$$FV = PMT \left( \frac{(1 + i)^n - 1}{i} \right)$$

$FV$  = Future Value,  $PMT$  = Payment,  $i$  = Periodic Rate,  $n$  = Number of periods made

### Present Value Annuities

Distinguishing feature: a series of payments where the balance decreases in value over time.

$$PMT = PV \left( \frac{i}{1 - (1 + i)^{-n}} \right)$$

$PV$  = Present Value,  $PMT$  = Payment,  $i$  = Periodic Rate,  $n$  = Number of periods remaining