Pivoting to perform Gauss-Jordan Reduction

Solve the system of linear equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 7 \\ -2x & + & 3y & - & z & = & 5 \\ -x & - & 2y & + & 3z & = & -1 \end{array}$$

Gaussian Reduction places a matrix into row-echelon form, but requires back-substitution. Gauss-Jordan Reduction places the matrix into reduced row-echelon form and does not require back-substitution.

$$\begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ -2 & 3 & -1 & 5 \\ -1 & -2 & 3 & -1 \end{array}\right] & \begin{array}{c} 13 \\ 5 \\ -1 \end{array} \end{array} \sim \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & & & \\ 0 & & & \end{array}\right] & \begin{array}{c} 13 \\ \\ \\ \end{array} \end{array} \longrightarrow \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 7 & 5 & 19 \\ 0 & 0 & 6 & 6 \end{array}\right] & \begin{array}{c} 13 \\ 31 \\ 12 \end{array} \end{array} \sim$$

✓ is sum of each row. Select any non-zero element on left side for the pivot element.

Rewrite pivot row. Clear pivot column. Pivot! (see below) Include ✓ column when pivoting.

Check ✓ by readding rows. Reduce any rows with a common factor.

$$\begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 7 & 5 & 19 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} 13 \\ 31 \\ 2 \end{array} \end{array} \sim \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} & & 0 & \\ 0 & & 0 & \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} \\ \\ 2 \end{array} \end{array} \longrightarrow \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} 7 \\ 21 \\ 2 \end{array} \end{array} \sim$$

Select new pivot element. You may only pivot once in any row or column.

Rewrite row, clear column. Previously cleared columns will remain cleared.

Pivot! Check the sums against the [column. Reduce any rows with a common factor.

$$\begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} 7 \\ 3 \\ 2 \end{array} \end{array} \sim \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} & 0 & 0 & \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} \\ 3 \\ 2 \end{array} \end{array} \longrightarrow \begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} 1 \\ 3 \\ 2 \end{array} \end{array}$$

Select new pivot element. Pivot on a 1 when possible. Only once per row or column. Since row 3 already has a 0 in the pivot column, we don't need to clear it.

Rewrite pivot row. Clear pivot column. Previously cleared columns will remain cleared. Rewrite any row with a 0 already in pivot column.

Pivot! Read solution. { (0, 2, 1) }

How to perform the actual pivot

$$\begin{array}{cc} \begin{array}{cccc} x & y & z & \text{rhs} \end{array} & \checkmark \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ -2 & 3 & -1 & 5 \\ -1 & -2 & 3 & -1 \end{array}\right] & \begin{array}{c} 13 \\ 5 \\ -1 \end{array} \end{array}$$

Repeat process for each element that is unknown in next matrix.

Take the difference of the products of the corners of the box formed by the pivot element and the element to be replaced. Always take the diagonal with the pivot element minus the diagonal without the pivot element.

New values for Row 2

y: 1(3) - (-2)(2) = 3 + 4 = 7
z: 1(-1) - (-2)(3) = -1 + 6 = 5
rhs: 1(5) - (-2)(7) = 5 + 14 = 19
✓: 1(5) - (-2)(13) = 5 + 26 = 31

New values for Row 3

y: 1(-2) - (-1)(2) = -2 + 2 = 0
z: 1(3) - (-1)(3) = 3 + 3 = 6
rhs: 1(-1) - (-1)(7) = -1 + 7 = 6
✓: 1(-1) - (-1)(13) = -1 + 13 = 12

Technology?

The TI-83 has two functions ref() and rref() that place a matrix into row-echelon form and reduced row-echelon form. Derive provides the row_reduce command. Unfortunately, neither one of these utilities will show work.

The instructor has written a pivot program for the TI-82, 83, and 85 calculators that will show intermediate matrices.