# Polynomial Functions

Polynomial functions in one variable can be written in expanded form as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

Examples of polynomials in expanded form are

$$f(x) = 5x^3 - 4x + 2 \text{ and } f(x) = -1.7x^8 + 3.1x^7 - 5x^4 + 3x - 2$$

Polynomials can be written in factored form as

$$f(x) = a_n (x - c_1)^{r_1} (x - c_2)^{r_2} \cdots (x - c_k)^{r_k}, \ r_1 + r_2 + \cdots r_k = n$$

where the $c$'s are the real or complex roots of the polynomial.
Examples of polynomials in factored form are

$$f(x) = -3x(x-2)^2(x+1) \text{ and } f(x) = x(x^2+3)^2(x-4)(x+5)$$

1. Polynomials only have variables with _______ ________ powers. That means no radicals or division involving the variable are possible.

2. The domain of a polynomial is all _______ __________.

3. In expanded form, the degree of the polynomial is the ___________ exponent.

4. In factored form, the degree of the polynomial is the _______ of the exponents. If any of the factors are non-linear, then use rules of exponents to determine the proper degree.

5. The leading term is the term in expanded form with the __________ exponent. In factored form, the leading terms of each factor need to be __________ together to find the leading term.

6. The leading coefficient is the ___________ of the leading term.

7. In expanded form, the constant is easy to spot. In factored form, the constant is found by ____________ the constants in each factor. Be sure to watch out for powers other than 1.

*Practice: Write the leading term, degree, leading coefficient, and constant for each polynomial.*

| Polynomial | Leading Term | Degree | Leading Coeff. | Constant |
|---|---|---|---|---|
| $f(x)=5x^6-4x^3+2x-5$ | | | | |
| $f(x)=-3x^4+5x^2+4$ | | | | |
| $f(x)=3(x-2)^2(x+5)(2x-3)$ | | | | |
| $f(x)=-4x(x^2+1)(x-3)(x+2)^3$ | | | | |

8. According to the Fundamental Theorem of Algebra, there is at least ___________ *real* or *complex* zero of a polynomial.

9. According to the Corollary to the Fundamental Theorem of Algebra, the number of *real* or *complex* zeros is equal to the ___________ of the polynomial, although the zeros may not be unique.

10. The maximum number of *real* roots is equal to the ___________ of the polynomial. The number of *real* roots may decrease by ___________.

11. The maximum number of extrema / turns (relative maximums and relative minimums) is equal to ___________ ___________ than the degree of the polynomial. The number of turns may decrease by ___________.

12. The right side of the graph (as $x \to +\infty$) will go ___________ ($y \to +\infty$) if the leading coefficient is ___________ and ___________ ($y \to -\infty$) if the leading coefficient is ___________.

13. The left side of the graph (as $x \to -\infty$) will do the ___________ as the right side if the degree is ___________ and the ___________ of the right side if the degree is ___________.

14. The *y* intercept of a polynomial is the ___________.

15. Polynomials are ______________, they can be drawn without lifting your pencil.

16. Polynomials are ______________, they have no sharp turns.

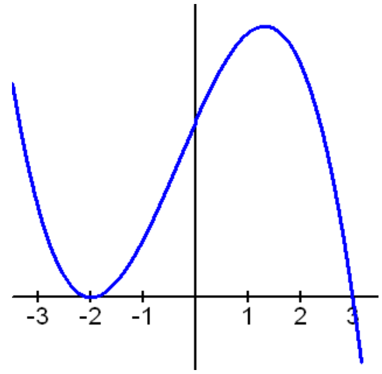
17. If $x = a$ is a root of a polynomial, then ___________ is a factor of the polynomial.

18. Roots are also called ___________ and ____________. *Real* roots (but not complex roots) are also called _____________.

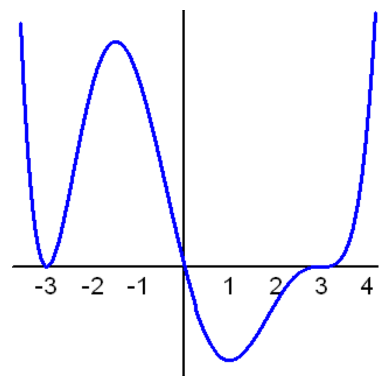
19. Assume that $x - a$ is a factor of $f(x)$ and that $x = a$ is a real number.

Graphs for #19

$f(x) = -(x-3)(x+2)^2$




$f(x) = x(x-3)^3(x+3)^2$




a. The graph will cross the *x*-axis and change sides at $x = a$ if the exponent on the factor is __________ .

b. The graph will touch the *x*-axis but stay on the same side at $x = a$ if the exponent on the factor is ___________.

c. The larger the exponent on a factor, the ___________ the graph will be near that root.

d. The only place that a polynomial (any continuous function) can change signs is at an ________________.

*Practice: Write a polynomial function, in factored form, that is negative on the far right side, crosses the x-axis at x=3, and touches the x-axis at x=-1.*

*Practice: Write a polynomial function, in factored form, that is positive on the far right side, touches the x-axis at x=4 and x=-4, and crosses the x-axis at x=0.*

20. Sign Charts

a. Look at the function in ______________ form and find all the zeros. Put these on number line in the proper order.

b. When making a sign chart, always start on the ___________ and work your way to the _____________. This is because the sign on the ___________ side can be determined by looking at the leading coefficient.

c. Look at the _____________ on the factor corresponding to each root and decide whether the sign on the left will be the same as the sign on the right or the opposite of the sign on the right.

*Practice: Make a sign chart for* $f(x) = -2(x-3)(x+4)^2(x+1)^3$

21. Descartes' Rule of Signs (requires real coefficients)

a. The maximum number of positive *real* roots is equal to the number of sign changes in _____________.

b. The maximum number of negative *real* roots is equal to the number of sign changes in _____________.

c. Either of these values may independently decrease by ___________, as long as they remain non-negative.

d. When finding the signs for $f(-x)$, only the signs on the ___________ powered terms will change, the signs on the ___________ powered terms will remain the same.

22. Factorization (requires real coefficients)

a. Every polynomial can be factored using _____________ factors only, but the factors may involve complex numbers.

b. Complex solutions always come in ____________. If $a+bi$ is a solution, then ____________ is also a solution.

c. If the coefficients are *rational*, then irrational solutions involving square roots always come in ____________. If $a+\sqrt{b}$ is a solution, then ______________ is also a solution.

d. Every polynomial can be factored using _____________ and _____________ _____________ factors.

23. Rational Root Theorem:

a. If a polynomial has *integer* coefficients, then any rational zero will be of the form of a factor of the ______________ over a factor of the ______________ ________________.

b. The Rational Root Theorem is an existence theorem, it does not _______________ that there will be any rational roots, it only says that if there are any, then they will be of the form indicated.

*Practice: List all possible rational factors of* $f(x)=3x^4-5x^3+2x-4$

*Practice: There is a special case when there is no constant. In that case, x is a factor of all the terms and x=0 is one real root. Factor out the x first and then use the constant that remains to list the remaining possible rational roots of*

$f(x)=2x^5-5x^3+4x^2+3x$

24. Polynomial Division.

a. Polynomial division is used to divide polynomials. It is performed similar to __________ _____________.

b. Much of the writing in polynomial division is redundant and can be eliminated. The technique used to do this is called _____________ _____________.

$$\begin{array}{r|l} & \phantom{x^3 + {}} x^2 + 2x - 2 \\ \hline x+3 & x^3 + 5x^2 + 4x - 2 \\ & \underline{x^3 + 3x^2} \\ & \phantom{x^3 + {}} 2x^2 + 4x \\ & \phantom{x^3 + {}} \underline{2x^2 + 6x} \\ & \phantom{x^3 + 2x^2 + {}} -2x - 2 \\ & \phantom{x^3 + 2x^2 + {}} \underline{-2x - 6} \\ & \phantom{x^3 + 2x^2 + -2x - {}} 4 \end{array}$$

25. Synthetic Division

a. Synthetic division needs a _____________ factor with a leading coefficient of _____.

| | | | | |
|---|---|---|---|---|
| −3 | 1 | 5 | 4 | −2 |
| | | −3 | 6 | 6 |
| | 1 | 2 | −2 | 4 |

b. Write down the coefficients in the dividend and leave a blank line below them. If there are any missing powers, you will need to write a _________ as a placeholder.

c. To the left of the values, write the root that corresponds to the factor. If $x-a$ is the factor, then write $a$ to the left.

d. The process begins by bringing the _____________ coefficient down.

e. The number in the bottom row is _____________ by the value to the left and this product is written in the next column.

f. The values in the next column are _____________ and the sum is written at the bottom of that column. These last two steps are repeated until the division is done.

g. When dividing synthetically by $x-a$, the last number in the bottom row is the _____________ and the numbers before that are the _____________. The exponents on the quotient are ___________ less than the exponents on the dividend.

*Practice. Use synthetic division.* $\left(3x^4-5x^3+4x^2-2x+2\right)\div\left(x-2\right)$

26. Remainder Theorem:

a. If a polynomial is divided by $x - a$, then the ______________ is $f(a)$.

b. Synthetic division can be used to ______________ a polynomial. To find $f(a)$, divide synthetically by $x - a$.

27. Factor Theorem:

a. A value is a root of a polynomial if and only if the remainder is __________.

b. If the remainder is zero, then $x - a$ is one factor of the polynomial. The other factor is the ______________ from the synthetic division.

28. Upper and Lower Bound Theorems: (requires real coefficients and a positive leading coefficient - factor a -1 out if leading coefficient is negative)

a. Upper bound: If synthetic division is performed with a ___________ value and all the coefficients in the quotient are _____________, then there are no *real* roots greater than that value, that value is an upper bound.

b. Lower bound: If synthetic division is performed with a ___________ value and the coefficients in the quotient _____________ in sign, then there are no *real* roots less than that value, that value is a lower bound.

c. _______ can be considered as either positive or negative as needed to make the theorem work.

d. Another way to state the upper bound theorem (so that you don't have to worry about the leading coefficient being negative) is that the value is an upper bound if all the values in the quotient are the __________ __________.

e. The first thing to do when checking for upper or lower bounds is to look at the value being evaluated. The value can only be an upper bound if it is ____________ and a lower bound if it is ____________.

29. Determine which of the following are upper bounds, lower bounds, or neither.

a. $x = -2$ is a ___________ bound because the value is ___________ and the signs _______________. There are no roots _____________ than $x = -2$. If $f(x) = 2x^3 + 3x^2 - x - 2$, then $f(-2) =$ ____________.

| | | | | |
|---|---|---|---|---|
| –2 | 2 | 3 | –1 | –2 |
| | | –4 | 2 | –2 |
| | 2 | –1 | 1 | –4 |

b. $x = 5$ is an ____________ bound because the value is ___________ and the signs are all the __________. There are no roots _____________ than $x = 5$. If $f(x) = 2x^3 - 3x^2 - 18x - 62$, then $f(5) =$ ____________.

| | | | | |
|---|---|---|---|---|
| 5 | 2 | –3 | –18 | –62 |
| | | 10 | 35 | 85 |
| | 2 | 7 | 17 | 23 |

c. $x = -2$ is a ___________ bound because the value is ___________ and the signs _______________. In this case, we need the ________ to be ___________. There are no roots _____________ than $x = -2$. If $f(x) = 2x^3 - 3x^2 - 14x - 5$, then $f(-2) =$ ____________.

| | | | | |
|---|---|---|---|---|
| –2 | 2 | –3 | –14 | –5 |
| | | –4 | 14 | 0 |
| | 2 | –7 | 0 | –5 |

d. $x = 3$ is not an upper bound or a lower bound. It is a ___________ value, but the signs are not all the __________. There may be roots _____________ than $x = 3$. If $f(x) = x^3 - 3x^2 - 4x + 8$, then $f(3) =$____________.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | –3 | –4 | 8 |
| | | 3 | 0 | –12 |
| | 1 | 0 | –4 | – 4 |

e. $x = 3$ is not an upper bound or a lower bound because it is ___________ but the signs aren't all the ___________. Alternating signs only apply when the value is _____________. $x = 3$ is a root because the _____________ is zero. There may be roots _____________ than $x = 3$.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | −4 | 5 | −6 |
| | | 3 | −3 | 6 |
| | 1 | −1 | 2 | 0 |

f. $x = -5$ is not an upper bound or a lower bound because the value is ___________ but the signs don't ____________. Having the numbers all the same sign applies only when the value is ______________. There may be roots _____________ than $x = -5$.

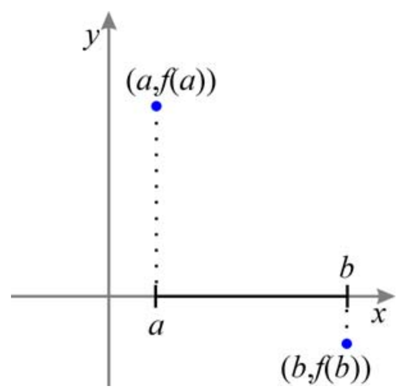
| | | | | |
|---|---|---|---|---|
| −5 | 1 | 8 | 15 | 7 |
| | | −5 | −15 | 0 |
| | 1 | 3 | 0 | 7 |

If $f(x) = x^3 + 8x^2 + 15x + 7$, then $f(-5) =$ ____________.

30. Intermediate Value Theorem:

a. A polynomial (any continuous) function will assume ___________ value between $f(a)$ and $f(b)$ on the interval $[a,b]$.

b. Of particular use to us is when $f(a)$ and $f(b)$ are different signs. In this case, there will be an ______________ of the function somewhere between $x=a$ and $x=b$.




Draw a function between the points without lifting your pencil. What has to happen on the interval?

31. Use the information given to answer the questions about the polynomial function.

$$f(x) = 6x^6 - 25x^5 + 38x^4 - 46x^3 + 44x^2 + 8x - 16$$
$$f(-x) = 6x^6 + 25x^5 + 38x^4 + 46x^3 + 44x^2 - 8x - 16$$
$$f(x) = (2x+1)(3x-2)(x-2)^2(x^2+2)$$

a. How many real or complex zeros are there?

b. What is the maximum number of extrema (turns)?

c. As $x \to +\infty$, what does *y* approach?

d. As $x \to -\infty$, what does *y* approach?

e. Any rational zeros will be a factor of ______ over a factor of ______.

f. What is the maximum number of positive real roots?

g. What is the maximum number of negative real roots?

h. List all real and complex zeros of the function.

i. Where will the graph of the function cross the x-axis?

j. Where will the graph of the function touch the x-axis?

k. What is the y-intercept of the graph of the function?

l. What is the domain of the function?

m. Make a sign chart for the function.

n. Sketch the graph of the function.

32. Use the information given to answer the questions about the polynomial function.

$$f(x) = -x^7 + 19x^5 - 99x^3 + 81x$$
$$f(-x) = x^7 - 19x^5 + 99x^3 - 81x$$
$$f(x) = -x(x^2 - 1)(x^2 - 9)^2$$

a. How many real or complex zeros are there?

b. What is the maximum number of extrema (turns)?

c. As $x \to +\infty$, what does $y$ approach?

d. As $x \to -\infty$, what does $y$ approach?

e. Any rational zeros will be a factor of ______ over a factor of ______.

f. What is the maximum number of positive real roots?

g. What is the maximum number of negative real roots?

h. List all real and complex zeros of the function.

i. Where will the graph of the function cross the x-axis?

j. Where will the graph of the function touch the x-axis?

k. What is the y-intercept of the graph of the function?

l. What is the domain of the function?

m. Make a sign chart for the function.

n. Sketch the graph of the function.

33. Putting it all together to find the roots of a polynomial.

Consider the polynomial equation $3x^4 + 7x^3 - 6x^2 - 12x + 8 = 0$

a. According to the Fundamental Theorem of Algebra, there are _________ real or complex roots.

b. According to Descartes' Rule of Signs, the maximum number of positive *real* roots is _________ and the maximum number of negative *real* roots is _________.

c. According to the Rational Root theorem, any *rational* roots will be of the form of a factor of _________ over a factor of _________.

When you list all of the possible rational roots in order, you get:

$$-8, -4, -\tfrac{8}{3}, -2, -\tfrac{4}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{4}{3}, 2, \tfrac{8}{3}, 4, 8$$

Perform synthetic division using a value from the list of possible rational roots. Begin with something simple and sort of in the middle. $x = 1$ works well.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 7 | –6 | –12 | 8 |
| | | 3 | 10 | 4 | –8 |
| | 3 | 10 | 4 | –8 | 0 |

d. Since we got a remainder of _______, that means that $x = 1$ is a ___________ and $x - 1$ is a ___________. We cannot use the ___________ bound theorem because the signs are not all the same.

We can factor the equation as $(x-1)(3x^3 + 10x^2 + 4x - 8) = 0$.

Now we try another factor, but we use the reduced polynomial $3x^3 + 10x^2 + 4x - 8$ instead of the original one. Let's try $x = 2$.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 10 | 4 | –8 |
| | | 6 | 32 | 72 |
| | 3 | 16 | 36 | 64 |

e. $x = 2$ is not a solution, but it is an ___________ bound, so we know not to try any of the values $x = \tfrac{8}{3}, 4, 8$. This is why we usually don't try extreme values like 8. They're unlikely to work and we are not able to eliminate any values if they don't.

Now we try something else. There's still a possibility of 1 positive from Descartes' Rule of Signs and we've ran out of nice values, so let's try a negative like $x = -1$.

$$\begin{array}{r|rrrr} -1 & 3 & 10 & 4 & -8 \\ & & -3 & -7 & 3 \\ \hline & 3 & 7 & -3 & -5 \end{array}$$

f. It didn't work and it's not a __________ bound, so let's move on and try something else, say $x = -2$.

$$\begin{array}{r|rrrr} -2 & 3 & 10 & 4 & -8 \\ & & -6 & -8 & 8 \\ \hline & 3 & 4 & -4 & 0 \end{array}$$

g. Woohoo! It worked. That means that $x = -2$ is a __________ and $x + 2$ is a __________.

We can now factor the equation as $(x-1)(x+2)(3x^2 + 4x - 4) = 0$

h. The other factor is a quadratic factor. When you get down to a quadratic factor, you can try __________, __________ __________, __________ the __________, or using the __________ __________ to find the remaining roots.

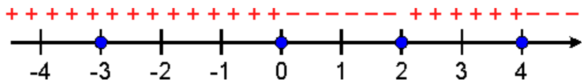
In this case, $3x^2 + 4x - 4$ factors as $(3x-2)(x+2)$.

i. The entire equation factors as $(x-1)(x+2)^2(3x-2) = 0$, so the roots are $x =$ _____, $x =$ _____, and $x =$ _____.

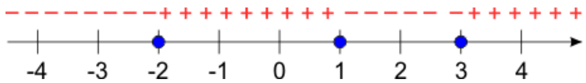
j. If you are performing synthetic division and you get zero as a remainder, it is possible that the value you used may work again. In this case, $x = -2$, was a solution twice. You may want to keep trying a root until you know it doesn't work or you can use the Rule of Signs to eliminate some answers.

34. Write a polynomial function that has the given sign chart.

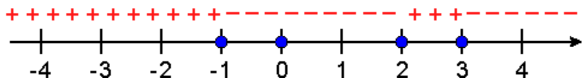
a.




b.

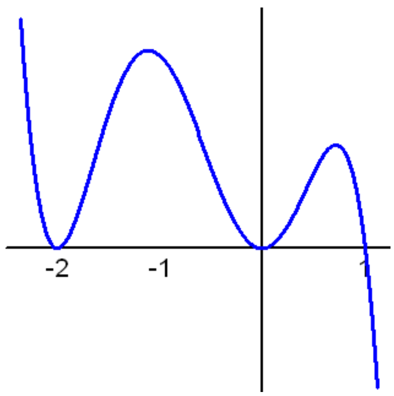



c.

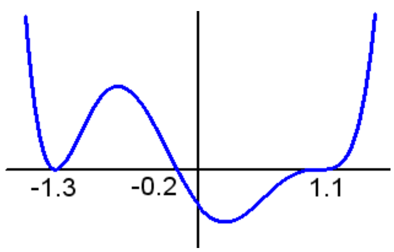



35. Write a function that has the graph shown.

a.




b.




c.