Rational Functions

Rational functions are the ratio of two polynomial functions.

They can be written in expanded form as

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\cdots+b_m x+b_0}$$

Examples of rational functions in expanded form are

$$f(x)=\frac{3x-2}{5x+7} \text{ and } f(x)=\frac{5x^3-3x^2+2x-4}{8x^3+4x-3}$$

Rational functions can also be written in factored form as

$$f(x)=\frac{a_n(x-c_1)^{p_1}(x-c_2)^{p_2}\cdots(x-c_j)^{p_j}}{b_m(x-d_1)^{q_1}(x-d_2)^{q_2}\cdots(x-d_k)^{q_k}}$$

$$p_1+p_2+\cdots+p_j=n,\ \ q_1+q_2+\cdots+q_k=m$$

where the *c*'s and *d*'s are real or complex numbers.
Examples of rational functions in factored form are

$$f(x)=\frac{x(2x-1)(x^2+1)}{(x-3)^2(x+4)} \text{ and } f(x)=\frac{(x-3)(x+4)^2}{(5x-1)(x+1)}$$

1. The domain is all real numbers except where the _____ is zero.

2. All information regarding roots or zeros of the function come from the _____.

3. The number of real or complex zeros is the _____ of the numerator.

4. Assume that $x=a$ is a real number and $x-a$ is a factor of the numerator ...

   a. The graph will cross the *x*-axis and change sides at $x=a$ if the exponent on the factor is _____.

   b. The graph will touch the *x*-axis but stay on the same side at $x=a$ if the exponent on the factor is _____.

5. All information regarding vertical asymptotes of the function come from the _____.

6. As *x* approaches a vertical asymptote, the *y* approaches _____ or _____.

7. Assume that $x = a$ is a real number and $x - a$ is a factor of the denominator ...

   a. The graph will be asymptotic in opposite directions to a vertical line at $x = a$ if the exponent on the factor is _____.

   b. The graph will be asymptotic in the same direction to a vertical line at $x = a$ if the exponent on the factor is _____.

8. The only places that a rational function can change from positive to negative or negative to positive are at an _____ or a _____ _____.

   a. The rational function will change signs and have a graph on different sides of the *x*-axis only if the exponent on the factor is _____.

   b. The rational function will stay the same sign and have a graph on the same side of the *x*-axis only if the exponent on the factor is _____.

Practice: Determine which values of x are x-intercepts and which are vertical asymptotes. For each x value, determine whether the function will change signs or stay the same sign on either side of these values.

Function	*x*-intercepts		vertical asymptotes	
	change	same	change	same
$f(x) = -\dfrac{2(x+1)(x-3)^2}{(x+4)^3(x-5)}$				
$f(x) = \dfrac{x^2(x-4)(x^2+4)}{(x-5)(x+3)^4}$				

9. The right and left hand behavior of the graph are determined by the degrees of the numerator and denominator.

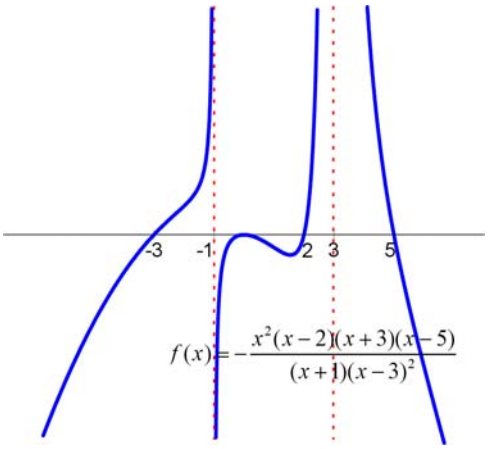
a. There will be no horizontal asymptote if there are more *x*'s in the _____.

b. The graph will have a horizontal asymptote at *y*=0 (the *x*-axis) if there are more *x*'s in the _____.

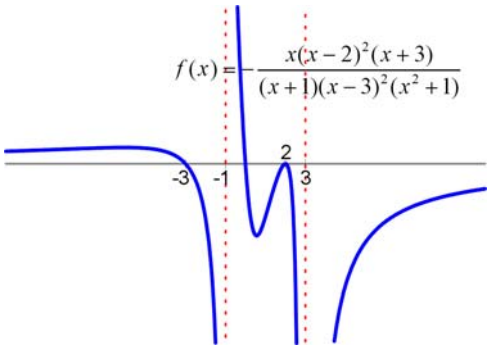
c. The graph will have a horizontal asymptote at the ratio of the leading coefficients if the degree in the numerator and the denominator are the _____.

d. The graph will have an oblique (slant) asymptote if there is exactly one more *x* in the _____. The slope of the oblique asymptote is the _____ of the leading coefficients.

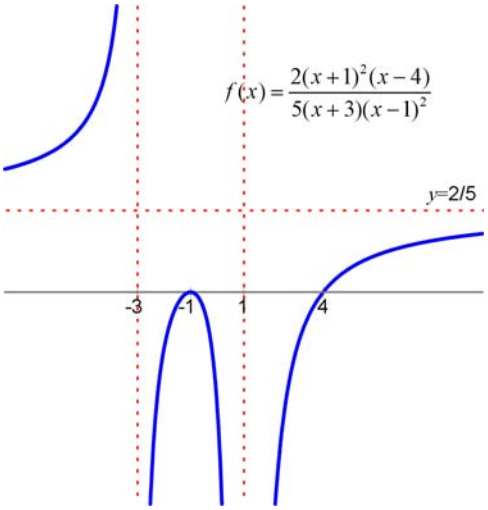
(a)



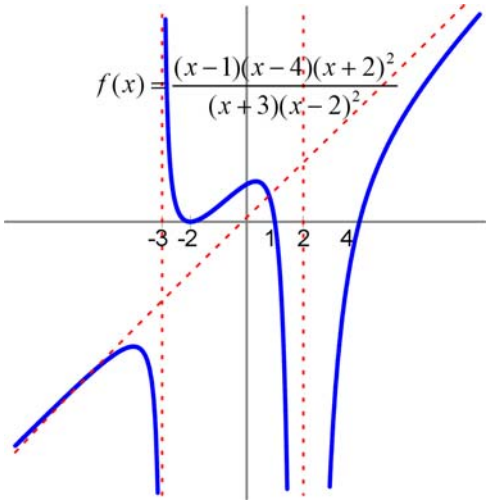
(b)



(c)



(d)



10. Horizontal asymptotes are only asymptotes as *x* approaches _____ or _____; that is, to the far _____ or the far _____ of the graph.

11. The function may cross the horizontal asymptote in the middle section of the graph. Basically, the middle section is anything between the smallest and largest _____ or _____ _____.

12. Although the graph may cross a horizontal asymptote, the graph can never cross a _____ _____.

13. As you graph the function, start on the _____ and work your way to the _____.

14. If there is a common factor between the numerator and the denominator, then there may or may not be a vertical asymptote at that value. Assume that $x-a$ is a common factor (the powers may be different) of both the numerator and denominator.

    a. There will be a vertical asymptote at $x=a$ if, after simplifying the rational function, $x-a$ is still in the _____.

    b. There will be a hole in the graph on the *x*-axis at $x=a$ if, after simplifying the rational function, $x-a$ is still in the _____. The value $x=a$ (is / is not) an *x*-intercept.

    c. There will be a hole in the graph, but not on the *x*-axis, at $x=a$ if, after simplifying the rational function, $x-a$ is no longer in _____ the numerator or the denominator.

    d. Regardless of where the $x-a$ ends up after simplification, it is important to note that $x=a$ is not in the _____ of the function because it originally caused division by _____.

Practice: Identify the x value associated with the common factor and determine if there will be an x-intercept, hole on the x-axis, hole off the x-axis, or vertical asymptote there.

$$f(x)=\frac{2(x-3)^2(x+4)}{4(x+3)(x+4)^2}$$

$$f(x)=-\frac{3(x+1)(x+4)}{5(x+1)(x-2)^2}$$

15. The *y*-intercept is the ratio of the _____ of the numerator and denominator.

16. Rational functions are _____, there are no sharp turns.

17. Rational functions are _____, they can be drawn without lifting up your pencil, except at _____ _____ or _____.

18. Complex roots aren't _____ or _____ _____ and the function cannot change signs at them.

19. Complex roots may need to be added so that the right and left hand behavior is correct. While it may not always give the correct graph, the simplest complex factor to insert into the rational function is _____.

20. Complex roots may also be necessary when there are extra _____ in the graph, just like they were for polynomial functions.

21. In the case where there is no horizontal asymptote, the right hand behavior (as $x \to +\infty$) is determined by looking at the _____ of the leading coefficients. Be sure to pay attention to whether or not there is a leading negative sign.

22. Even if there is a horizontal or oblique asymptote, you can still determine whether the right hand side is _____ or _____ by looking at the sign of the ratio of the leading coefficients.

23. The left hand behavior (as $x \to -\infty$) is similar to that of polynomial functions, except that instead of looking at the degree of the polynomial, you should look at the _____ in the degrees of the numerator and denominator.

    a. If the difference of the degrees is _____, then the graph will be on the same side of the horizontal asymptote as the right side.

    b. If the difference of the degrees is _____, then the graph will be on opposite sides of the horizontal asymptote as the right side.

24. For large (postive or negative) values of *x*, only the _____ _____ matter. All other powers of *x* are insignificant in comparison.

25. Answer the questions about the rational function (be sure to note the - in front).

$$f(x) = -\frac{8x^3(x-2)(x+4)^2(x-5)}{5(x-2)(x+1)^2(x-3)^3(x+3)}$$

a. What is the domain of the function f?

b. Simplify the function, stating any necessary restrictions.

c. Check the box in each row that describes the graph of the function at the indicated value of x.

value	hole, not on x-axis	hole on x-axis	vertical asymptote	regular x-intercept	none of these
$x = 5$					
$x = 2$					
$x = 3$					
$x = -1$					
$x = -4$					

d. Which of the following statements describes the graph of the function? (Circle one and fill in the blank if appropriate)

i. There is no horizontal asymptote.
ii. There is an oblique asymptote with slope $m =$ _____.
iii. The horizontal asymptote is the x-axis ($y = 0$).
iv. There is a horizontal asymptote at the line $y =$ _____.

e. Make a sign chart for the function.

f. Sketch the graph of the function.

26. Answer the questions about the rational function.

$$f(x)=\frac{17\left(x^2-4\right)\left(x-3\right)^2}{3\left(x^2-1\right)^2\left(x^2-9\right)}$$

a. What is the domain of the function f?

b. Simplify the function, stating any necessary restrictions.

c. Check the box in each row that describes the graph of the function at the indicated value of x.

value	hole, not on x-axis	hole on x-axis	vertical asymptote	regular x-intercept	none of these
$x=1$					
$x=2$					
$x=3$					
$x=-3$					
$x=-4$					

d. Which of the following statements describes the graph of the function? (Circle one and fill in the blank if appropriate)

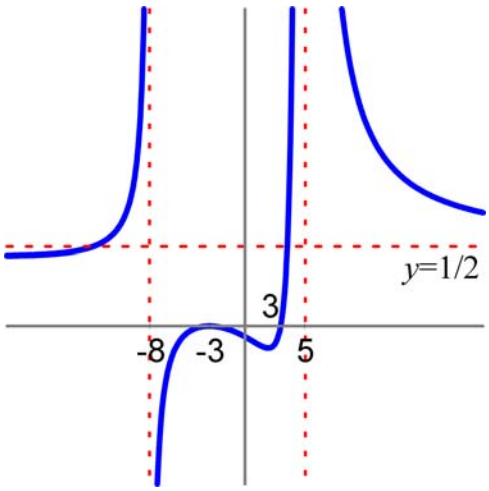
i. There is no horizontal asymptote.
ii. There is an oblique asymptote with slope m = _____.
iii. The horizontal asymptote is the x-axis (y = 0).
iv. There is a horizontal asymptote at the line y = _____.

e. Make a sign chart for the function.

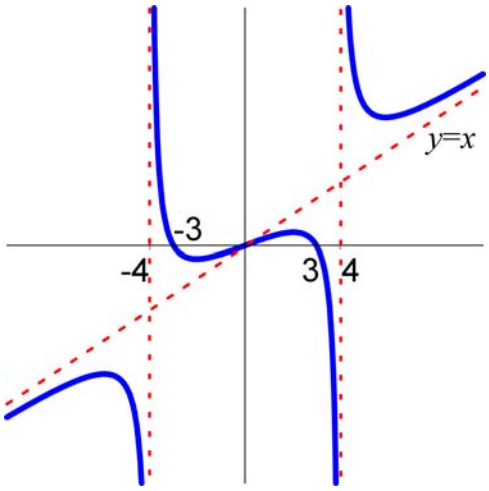
f. Sketch the graph of the function.

27. Write a function whose graph is shown.

a.



b.



c.