Math 230: Mathematical Notation

Purpose:
One goal in any course is to properly use the language of that subject. Differential Equations is no different and may often seem like a foreign language. These notations summarize some of the major concepts and more difficult topics of the unit. Typing them helps you learn the material while teaching you to properly express mathematics on the computer. Part of your grade is for *properly* using mathematical content.

Instructions:
Use Word or WordPerfect to recreate the following documents. Each article is worth 10 points and should be emailed to the instructor at james@richland.edu. This is not a group assignment, each person needs to create and submit their own notation.

Type your name at the top of each document. Include the title as part of what you type.

For expressions or equations, you should use the equation editor in Word or WordPerfect. Note that the equation editor in recent versions of Microsoft Word is not as powerful as older versions. You may want to use Insert / Object / Microsoft Equation 3.0 instead. Another option is to install MathType and use it.

If there is an equation, put both sides of the equation into the same equation editor box instead of creating two objects. Be sure to use the proper symbols, there are some instances where more than one symbol may look the same, but they have different meanings and don't appear the same as what's on the assignment. There are some useful tips on the website at http://people.richland.edu/james/editor/

If you fail to type your name on the document, you will lose 1 point.

These notations are due before the beginning of class on the day of the exam for that material. For example, notation 3 is due on the day of the chapter 3 exam. Late work will be accepted but will lose 20% of its value per class period.

Chapter 1 - Introduction to Differential Equations

A **linear** differential equation is one where all occurrences of the dependent variable and its derivatives are raised to the first power. The **order** of a differential equation is the order of the highest derivative in the equation.

A differential equation will have an unique solution if both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on some region.

Mathematical Models

Population Dynamics: The rate of population growth is proportional to the total population at that time. $dP/dt = kP$

Radioactive Decay: The rate at which the nuclei of a substance decay is proportional to the number of nuclei remaining. $dA/dt = kA$

Newton's Law of Cooling: The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the surrounding medium. $dT/dt = k(T - T_m)$

Chemical Reactions: The rate at which a reaction proceeds is proportional to the product of the remaining concentrations. $dX/dt = k(\alpha - X)(\beta - X)$

Series Circuits: Kirchoff's second law says $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

Falling Bodies: Without air resistance and a positive upwards direction, $m\frac{d^2s}{dt^2} = -mg$ or $m\frac{dv}{dt} = -mg$. With air resistance (viscous damping) and a positive downward direction, $m\frac{dv}{dt} = mg - kv$ or $m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg$.

Slipping Chain: For a chain in motion around and frictionless peg, $\frac{d^2x}{dt^2} - \frac{2g}{L}x = 0$

Suspended Cables: If T_1 is the tension tangent to the lowest point and W is the portion of the vertical load between two points, then $dy/dx = W/T_1$

Chapter 2 - First-Order Differential Equations

A differential equation is **autonomous** if it is a function of the dependent variable only .

A first-order DE is **separable** if it can be written in the form $dy/dx = g(x)h(y)$.

The standard form for a **linear** first-order DE is $dy/dx + P(x)y = f(x)$ and is homogeneous if $f(x) = 0$. The solution to this DE is the sum of two solutions $y = y_c + y_p$ where y_c is the general solution to the homogenous DE and y_p is the particular solution to the nonhomogeneous DE. The procedure known as variation of parameters leads to an integrating factor $\mu = e^{\int P(x)dx}$.

The error function and complementary error functions are defined by $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$, where $erf(x) + erfc(x) = 1$.

For a function $z = f(x,y)$, the differential is $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. If the function is a constant, then the differential is 0. A DE of the form $M(x,y)dx + N(x,y)dy = 0$ is an **exact** differential equation if the left hand side is a differential of some function $f(x,y)$. If M and N are continuous and have continuous partial derivatives on some region, then it is exact if and only if $N_x = M_y$. If a DE is exact, then you can find the potential function $f(x,y)$ by integrating $\int Mdx$ and $\int Ndy$ and finding the union of all the terms.

A **function is homogeneous** of degree α if it has the property that $f(tx,ty) = t^\alpha f(x,y)$. The substitutions $y = ux$ or $x = vy$ will reduce a homogeneous equation to a separable first-order DE.

Bernoulli's equation is $dy/dx + P(x)y = f(x)y^n$ and can be solved with the substitution $u = y^{1-n}$.

Use **ASLEHBN** as a way to remember the order of attacking a first-order DE: Autonomous, Separable, Linear, Exact, Homogenous, Bernoulli, or none of these.

Chapter 3 - Modeling with First-Order Differential Equations

Falling Body
The model for a falling body where air resistance is proportional to the velocity is $dv/dt = g - \frac{k}{m}v$.

Kirchoff's Laws
Let $E(t)$ be impressed voltage, $i(t)$ be current, $q(t)$ be charge, L be inductance, R be resistance, and C be capacitance. Current and charge related by $i(t) = dq/dt$.

Conservation of Charge (1st law): The sum of the currents entering a node must equal the sum of the currents exiting a node.

Conservation of Energy (2nd law): The voltages around a closed path in a circuit must sum to zero (voltage drops are negative, voltage gains are positive).
The voltage drop across an inductor is $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$. The voltage drop across a resistor is $iR = R\frac{dq}{dt}$. The voltage drop across a capacitor is $\frac{1}{C}q$. The sum of the voltage drops is equal to the impressed voltage $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$.

Logistic Equation
When the rate of growth of a population P is proportional to the amount present and the amount remaining before reaching the carrying capacity L, then the resulting DE is $dP/dt = kP(L-P)$.

Lotka-Volterra Predator-Prey Model
If $x(t)$ is the population of a predator and $y(t)$ is the population of the prey at time *t*, then the populations can be modeled by the system of nonlinear system of DEs: $dx/dt = x(-a+by)$ and $dy/dt = y(d-cx)$.

Chapter 4 - Higher-Order Differential Equations

Superposition Principle - Homogeneous Equations: A linear combination of solutions to a homogeneous DE is also a solution. This means that constant multiples of a solution to a homogeneous DE are also solutions and the trivial solution $y=0$ is always a solution to a homogeneous DE.

A set of functions is linearly dependent if there is some linear combination of the functions that is zero for every *x* in the interval.

A set of solutions is linearly independent if and only if the Wronskian is not zero for every *x* in some interval. A set of linearly independent solutions to a homogeneous DE is set to be a fundamental set of solutions and there is always a fundamental set for a homogeneous DE.

The Wronskian is defined by $W\left(y_1, y_2, \cdots, y_n\right) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

Any function free of arbitrary parameters that satisfies a nonhomogeneous DE is a particular solution, y_p. The complementary function, y_c, is the general solution to the associated homogeneous DE. The general solution to a nonhomogeneous equation is $y = y_c + y_p$.

Reduction of Order: If $y_1(x)$ is a solution to a second-order linear homogeneous DE in standard form $y'' + P(x)y' + Q(x)y = 0$, then a second solution is $y_2(x) = y_1(x)\int \frac{\mu^{-1}}{y_1^2(x)}dx$, where $\mu = e^{\int P(x)dx}$ is the integrating factor from chapter 2.

Homogeneous Linear Equations with Constant Coefficients: The auxiliary equation is formed by converting the DE into a polynomial function. For example, $3y^{(5)} - 20y^{(4)} + 78y''' - 134y'' + 99y' - 26y = 0$ would have an auxiliary equation of $3m^5 - 20m^4 + 78m^3 - 134m^2 + 99m - 26 = 0$. You find the solutions to the auxiliary equation, which in this case are $m=1$ with multiplicity 2, $m = 2/3$, and $m = 2 \pm 3i$. From each of the roots, we form a linear independent combination of terms involving e.

Thus $y = c_1 e^x + c_2 x e^x + c_3 e^{\frac{2}{3}x} + e^{2x}\left(c_4 \cos 3x + c_5 \sin 3x\right)$.

Two common DEs $y'' + k^2 y = 0$ and $y'' - k^2 y = 0$ have solutions of $y = c_1 \cos kx + c_2 \sin kx$ and $y = c_1 e^{kx} + c_2 e^{-kx}$ respectively. The solutions to $y'' - k^2 y = 0$ can also be written as $y = c_1 \cosh kx + c_2 \sinh kx$.

Method of Undetermined Coefficients - Superposition Approach: This method is useful when the coefficients of the DE are constants and the input function is comprised of sums or products of constant, polynomial, exponential, or trigonometric (sine and cosine) functions. You make guesses about the particular solutions based on the form of the input and then equate coefficients.

Method of Undetermined Coefficients - Annihilator Approach: *L* is an annihilator of a function if it has constant coefficients and $L\left(f\left(x\right)\right) = 0$. In each case below, *k* is a whole number less than *n*.

- Use D^n to annihilate functions of the form x^k.
- Use $\left(D - \alpha\right)^n$ to annihilate functions of the form $x^k e^{ax}$.
- Use $\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]^n$ to annihilate functions of the form $x^k e^{ax} \cos \beta x$ or $x^k e^{ax} \sin \beta x$

Variation of Parameters: Variation of parameters can be used when the coefficients of the DE are not constants. It involves the Wronskian, W, and two functions $u_1' = -\frac{y_2 f\left(x\right)}{W}$ and $u_2' = \frac{y_1 f\left(x\right)}{W}$ that are integrated to find u_1 and u_2. The particular solution is then $y_p = u_1 y_1 + u_2 y_2$.

Cauchy-Euler Equation: A linear differential equation composed of terms $a_x x^k \frac{d^k y}{dx^k}$, where the a_k factors are constant, can be solved by trying $y = x^m$. Treat it like the auxiliary equation, except use $\ln x$ instead of *x*. For example, if the solutions are $m = 2 \pm 3i$, then $y = e^{2(\ln x)}\left[c_1 \cos\left(3 \ln x\right) + c_2 \sin\left(3 \ln x\right)\right]$, which simplifies to $y = x^2\left[c_1 \cos\left(3 \ln x\right) + c_2 \sin\left(3 \ln x\right)\right]$.

Chapter 5 - Modeling with Higher-Order Differential Equations

Free Undamped Motion: $m\frac{d^2x}{dt^2} = -kx$ can be written as $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$ and has a solution of $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

Free Damped Motion: $m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$ can be written as $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$ where $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$. Let $d = \lambda^2 - \omega^2$.

- If $d > 0$, the overdamped system has the solution $x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{d}t} + c_2 e^{-\sqrt{d}t}\right)$
- If $d = 0$, the critically damped system has the solution $x(t) = e^{-\lambda t}\left(c_1 + c_2 t\right)$.
- If $d < 0$, the underdamped system has the solution $x(t) = e^{-\lambda t}\left(c_1 \cos\left(t\sqrt{d}\right) + c_2 \sin\left(t\sqrt{d}\right)\right)$

Driven Motion: In driven motion, an external force $f(t)$ is applied to the system and the DE is $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$ where $F(t) = \frac{f(t)}{m}$. Use the method of undetermined coefficients or variation of parameters to solve the nonhomogeneous equation.

Series Circuit Analogue: The DE $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ is overdamped, critically damped, or underdamped depending on the value of the discriminant $R^2 - 4L/C$.

Deflection of a Beam: Deflection satisfies the DE $EI\frac{d^4y}{dx^4} = w(x)$ where EI is the flexural rigidity and $w(x)$ is the load per unit length.

- An embedded end has $y = 0$ and $y' = 0$
- A free end has $y'' = 0$ and $y''' = 0$
- A simply supported end has $y = 0$ and $y'' = 0$

Chapter 6 - Series Solutions of Linear Equations

If $x = x_0$ is an ordinary point, then a power series centered at x_0 is $y = \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^k$.

Method of Frobenius: If $x = x_0$ is a regular singular point then there exists at least one solution of the form $y = \left(x - x_0\right)^r \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^k$ which simplifies to

$y = \sum_{k=0}^{\infty} c_k \left(x - x_0\right)^{k+r}$, where *r* is a constant to be determined.

Bessel's Equation of Order *v*: $x^2 y'' + xy' + \left(\alpha^2 x^2 - v^2\right) y = 0$

The solution is $y = c_1 J_v\left(\alpha x\right) + c_2 J_{-v}\left(\alpha x\right)$ as long as *v* is not an integer.

If *v* is integer then the solution is $y = c_1 J_v\left(\alpha x\right) + c_2 Y_v\left(\alpha x\right)$. Technically, this is a solution to any Bessel's equation, but we prefer J_v and J_{-v} when v is not an integer.

Modified Bessel Equation:

The solution to $x^2 y'' + xy' - \left(\alpha^2 x^2 + v^2\right) y = 0$ is $y = c_1 I_v\left(\alpha x\right) + c_2 K_v\left(\alpha x\right)$. If *v* is not an integer, then you can use I_v and I_{-v} like we did with the Bessel Equation.

Legendre's Equation of Order *n*: $\left(1 - x^2\right) y'' - 2xy' + n\left(n+1\right) y = 0$. If *n* is a non-negative integer, then $P_n\left(x\right)$ is the solution and some of the solutions are $P_0\left(x\right) = 1$, $P_1\left(x\right) = x$, $P_2\left(x\right) = \frac{1}{2}\left(3x^2 - 1\right)$, $P_3\left(x\right) = \frac{1}{2}\left(5x^3 - 3x\right)$, and

$P_4\left(x\right) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$. If *n* is not a non-negative integer, then the solution is an infinite series.

Chapter 7 - The Laplace Transform

Let f be a function defined for $t \geq 0$. The Laplace transform of $f(t)$ is $F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, provided this integral converges.

Laplace transform of a derivative:
$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

First Translation Theorem: $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$

Unit Step Function: Also known as the Heaviside function, it is useful for creating piecewise functions. $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$

Second Translation Theorem: If $a > 0$ then $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as} F(s)$

Derivatives of Transforms: $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

Convolution: Convolutions, defined by $f * g = \int_0^t f(\tau) g(t-\tau)\,d\tau$ are commutative, $f * g = g * f$, and the Laplace transform of a convolution is the product of the Laplace transforms, $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$. If you let $g(t) = 1$, then the transform of an integral is $\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$.

Transform of a Periodic Function: $\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$, where the period is *T*.

Dirac Delta Function: $\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)$ is ∞ when $t = t_0$ and 0 otherwise. $\int_0^\infty \delta(t-t_0)\,dt = 1$ and $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$.

Chapter 8 - Systems of Linear First-Order DEs

Eigenvalues and Eigenvectors: If $\mathbf{X}' = \mathbf{AX}$ is a homogeneous linear first-order system, then the polynomial equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is the characteristic equation and its solutions are the eigenvalues. We want to write a solution as $\mathbf{X} = \mathbf{K}e^{\lambda t}$ where $\mathbf{K}$ is the associated eigenvector.

The general solution to a homogenous linear system is
$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}$

If $\mathbf{K}$ is an eigenvector corresponding to the complex eigenvalue $\lambda = \alpha + \beta i$, then let $\mathbf{B}_1 = \operatorname{Re}(\mathbf{K})$ and $\mathbf{B}_2 = \operatorname{Im}(\mathbf{K})$. The two solutions with real coefficients are $\mathbf{X}_1 = [\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t]e^{\alpha t}$ and $\mathbf{X}_2 = [\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t]e^{\alpha t}$.

For a nonhomogeneous system, the general solution becomes $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ and the method of undetermined coefficients or variation of parameters can be used to find the particular solution.

Matrix Exponentials: For a homogeneous system, we can define a matrix exponential $e^{\mathbf{A}t}$ so that $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution to $\mathbf{X}' = \mathbf{AX}$. For any square matrix of size *n*, $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots$, which can be written as $e^{\mathbf{A}t} = \sum_{k=0}^{\infty}\mathbf{A}^k\frac{t^k}{k!}$. $e^{\mathbf{A}t}$ is a fundamental matrix.

For nonhomogeneous systems, $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, the general solution is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds$. In practice, $e^{-\mathbf{A}s}$ can be found from $e^{\mathbf{A}t}$ by substituting $t = -s$.

Chapter 9 - Numerical Solutions of Ordinary DEs

Euler's Method: In chapter 2 (and in Calculus II), we had Euler's Method, where $y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Improved Euler's Method: This method estimates the next *y* value in the sequence using Euler's method, $y^*_{n+1} = y_n + h \cdot f(x_n, y_n)$, and then uses that estimate in a midpoint formula to find the next *y* used. $y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$

Runge-Kutta Methods: These are generalizations of Euler's method where the slope $f(x_n, y_n)$ is replaced by a weighted average of the slopes on the interval $x_n \le x \le x_{n+1}$. That is, $y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + \cdots + w_mk_m)$, where the weights *w* are chosen so that they agree with a Taylor series of order *m*.

RK1: The first-order Runge-Kutta method is actually Euler's method. Choose $k_1 = f(x_n, y_n)$ and $w_1 = 1$ to get $y_{n+1} = y_n + h \cdot f(x_n, y_n)$.

RK2: The second-order Runge-Kutta method chooses values $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + h, y_n + hk_1)$, and $w_1 = w_2 = \frac{1}{2}$ to get the improved Euler's method where $y_{n+1} = y_n + h\left[\frac{1}{2} f(x_n, y_n) + \frac{1}{2} f(x_n + h, y_n + hk_1)\right]$

RK4: Let $w_1 = w_4 = \frac{1}{6}$ and $w_2 = w_3 = \frac{1}{3}$. Choose $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$, $k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$, and $k_4 = f(x_n + h, y_n + hk_3)$.