MATH 230 - Mathematical Notation

Introduction

Purpose

One goal in any course is to properly use the language of that subject. Differential Equations is no different and may often seem like a foreign language. These notations summarize some of the major concepts and more difficult topics of the unit. Typing them helps you learn the material while teaching you to properly express mathematics on the computer. Part of your grade is for properly using mathematical content.

This is not a group assignment, each person needs to create and submit their own notation. If it is determined that sharing occurs, a reduced grade will result for all parties involved.

Instructions

Use Microsoft Word to recreate the following documents and submit them through the Canvas learning management system. Save the Word documents as .docx format; do not send .rtf files.

There will be one document for each exam that contains the sections of this document for the covered chapters. For example, the first test is over chapters 1 and 2, so the notation for chapters 1 and 2 should be typed in a single document and submitted before the chapter 1 and 2 exam. Late work will be accepted but will lose 20% of its value per class period.

Place your name in the heading of the document so that it appears on each page. Failure to put your name in the document will cost you points. Include the chapter titles as part of what you type. You do not have to exactly match the formatting (fonts, line breaks, etc.,) inside this document.

Regular typing should be done with the word processing software, but when you come to mathematical content, you should use the equation editor in Word. Note that the equation editor in recent versions of Microsoft Word is not as powerful as older versions and you may wish to use *Insert / Object / Microsoft Equation 3.0* instead. Another option is to install MathType and use it (they have a 30 day free trial).

When there is mathematical content, put the entire item into a single equation object instead of creating multiple objects. Don't just put the portion that can't be typed directly into an object (to get x^2, don't type x and then use an object for the squared). Be sure to use the proper symbols, there are some instances where more than one symbol may look the same, but they have different meanings and don't appear the same as what's on the assignment. There are some useful tips at https://people.richland.edu/james/editor/

Boxes like this one contain hints for typing the document. Do not include these boxes in your document.

Grading Rubric

Criteria	**Ratings**		
Mathematics	Minor issues 4 pts	Major issues 2 pts	No objects 0 pts
Completeness	Missing few words 4 pts	Missing sentences 2 pts	Missing large portion 0 pts
Formatting	Has both name and title 2 pts	Missing name or title 1 pt	Missing both 0 pts

Chapter 1 - Introduction to Differential Equations

A linear differential equation is one where all occurrences of the dependent variable and its derivatives are raised to the first power. The order of a differential equation is the order of the highest derivative in the equation.

A differential equation $\frac{dy}{dx} = f(x,y)$ will have an unique solution if both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on some region.

Notation

There are four types of notation that we will use for ordinary derivatives in this course.

- Leibniz: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\ldots$, $\frac{d^ny}{dx^n}$
- Lagrange: y', y'', y''', $\ldots$, $y^{(n)}$
- Euler: Dy, D^2y, D^3y, $\ldots$, D^ny (used in chapter 4)
- Newton: $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$ (used when derivatives are with respect to time)

Mathematical Models

These models are presented here, but the development and application are covered in later chapters.

First-Order Linear Models

Population Dynamics: The rate of population growth is proportional to the total population at that time. $\frac{dP}{dt} = kP$.

Radioactive Decay: The rate at which the nuclei of a substance decay is proportional to the number of nuclei remaining. $\frac{dA}{dt} = kA$

Newton's Law of Cooling: The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the surrounding medium. $\frac{dT}{dt} = k(T - T_m)$

Second-Order Linear Models

Series Circuits: Kirchhoff's second law says $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

Falling Bodies: Without air resistance and a positive upwards direction, $m\frac{d^2s}{dt^2} = -mg$ or $m\frac{dv}{dt} = -mg$. With air resistance (viscous damping) and a positive downward direction, $m\frac{dv}{dt} = mg - kv$ or $m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg$.

Non-Linear Models

Chemical Reactions: The rate at which a reaction proceeds is proportional to the product of the remaining concentrations. $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$

Spread of Disease: The rate at which a disease spreads is jointly proportional to the number of people who have been exposed and the number of people who haven't been exposed. $\frac{dP}{dt} = kP(L - P)$

Chapter 2 - First-Order Differential Equations

A differential equation is **autonomous** if it is a function of the dependent variable only .

A first-order DE is **separable** if it can be written in the form $\frac{dy}{dx} = g(x)h(y)$.

The **standard form** for a **linear** first-order DE is $\frac{dy}{dx} + P(x)y = f(x)$ and is **homogeneous** if $f(x) = 0$. The solution to this DE is the sum of two solutions $y = y_c + y_p$ where y_c is the general solution to the homogeneous DE and y_p is the particular solution to the nonhomogeneous DE. The procedure known as variation of parameters leads to an integrating factor $\mu = e^{\int P(x)dx}$.

For a function $z = f(x,y)$, the **differential** is $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. If the function is a constant, then the differential is 0. A DE of the form $M(x,y)dx + N(x,y)dy = 0$ is an **exact** differential equation if the left hand side is a differential of some function $f(x,y)$. If M and N are continuous and have continuous partial derivatives on some region, then it is exact if and only if $N_x = M_y$. If a DE is exact, then you can find the potential function $f(x,y)$ by integrating $\int M\,dx$ and $\int N\,dy$ and collecting the distinct terms.

A **function is homogeneous** of degree α if it has the property that $f(tx,ty) = t^\alpha f(x,y)$. If both M and N are homogeneous functions of the same degree, then the substitutions $y = ux$ or $x = vy$ will reduce $M\,dx + N\,dy = 0$ to a separable first-order DE.

Bernoulli's equation is $\frac{dy}{dx} + P(x)y = f(x)y^n$ and can be solved with the substitution $u = y^{1-n}$.

Use **ASLEHBN** as a way to remember the order of attacking a first-order DE: Autonomous, Separable, Linear, Exact, Homogeneous, Bernoulli, or None of these.

Chapter 4 - Higher-Order Differential Equations

Superposition Principle - Homogeneous Equations: A linear combination of solutions to a homogeneous DE is also a solution. This means that constant multiples of a solution to a homogeneous DE are also solutions and the trivial solution $y = 0$ is always a solution to a homogeneous DE.

A set of functions is linearly dependent if there is some linear combination of the functions that is zero for every x in the interval.

A set of solutions is linearly independent if and only if the Wronskian is not zero for every x in some interval. A set of linearly independent solutions to a homogeneous DE is said to be a fundamental set of solutions and there is always a fundamental set for a homogeneous DE.

The Wronskian of n functions $y_1, y_2, \ldots, y_n$, is the $n \times n$ determinant $W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

Any function free of arbitrary parameters that satisfies a nonhomogeneous DE is a particular solution, y_p. The complementary function, y_c, is the general solution to the associated homogeneous DE. The general solution to a nonhomogeneous equation is $y = y_c + y_p$.

Reduction of Order: If y_1 is a solution to a second-order linear homogeneous DE in standard form $y'' + P(x)y' + Q(x)y = 0$, then a second solution is $y_2 = y_1 \int \frac{\mu^{-1}}{y_1^2}dx$, where $\mu = e^{\int P(x)\,dx}$ is the integrating factor from chapter 2.

Homogeneous Linear Equations with Constant Coefficients: The auxiliary equation is formed by converting the DE into a polynomial function. For example, $3y^{(5)} + y^{(4)} + 3y''' + 109y'' + 192y' + 52y = 0$ would have an auxiliary equation of $3m^5 + m^4 + 3m^3 + 109m^2 + 192m + 52 = 0$. You find the solutions to the

auxiliary equation, which in this case are $m=-2$ with multiplicity 2, $m=-\frac{1}{3}$, and $m=2\pm 3i$. From each of the roots, we form a linear independent combination of terms involving e^{mx}. Thus
$y=c_1e^{-2x}+c_2xe^{-2x}+c_3e^{-1/3x}+e^{2x}(c_4\cos 3x+c_5\sin 3x)$.

Two common DEs $y''+k^2y=0$ and $y''-k^2y=0$ have solutions of $y=c_1\cos kx+c_2\sin kx$ and
$y=c_1e^{kx}+c_2e^{-kx}$ respectively. The solutions to $y''-k^2y=0$ can also be written as $y=c_1\cosh kx+c_2\sinh kx$.

Method of Undetermined Coefficients - Superposition Approach: This method is useful when the coefficients of the DE are constants and the input function is comprised of sums or products of constant, polynomial, exponential, or trigonometric (sine and cosine) functions. You make guesses about the particular solutions based on the form of the input and then equate coefficients.

Method of Undetermined Coefficients - Annihilator Approach: L is an annihilator of a function if it has constant coefficients and $L(f(x))=0$.

- Use D^n to annihilate functions of the form x^k, where k is a whole number less than n.
- Use $D-\alpha$ to annihilate functions of the form $e^{\alpha x}$.
- Use $D^2+\beta^2$ to annihilate functions for the form $\cos\beta x$ or $\sin\beta x$.
- Combinations of polynomial, exponential, and trigonometric functions can be annihilated by using $\left[(D-\alpha)^2+\beta^2\right]^n$ that annihilates functions of the form $x^ke^{\alpha x}\cos\beta x$ or $x^ke^{\alpha x}\sin\beta x$, where k is a whole number less than n.

Variation of Parameters: Variation of parameters can be used when the coefficients of the DE are not constants. It involves the Wronskian, W, and two functions $u_1'=-\frac{y_2f(x)}{W}$ and $u_2'=\frac{y_1f(x)}{W}$ that are integrated to find u_1 and u_2. The particular solution is then $y_p=u_1y_1+u_2y_2$.

Cauchy-Euler Equation: This looks like a linear DE with constant coefficients except that the coefficients have an extra factor of x^k, where k is the same as the order of the derivative. The solution is $y=x^m$, which makes $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$, and so on. Substitute and solve for m, but then replace x by $\ln x$ when writing the solution. For example, $x^2y''+5xy'+4y=0$ turns into $m(m-1)+5m+4=0$, which has a solution of $m=-2$ with multiplicity 2. The solution is $y=c_1e^{-2\ln x}+c_2(\ln x)e^{-2\ln x}$, which simplifies to
$y=c_1x^{-2}+c_2x^{-2}\ln x$.

Chapter 3 - Modeling with First-Order Differential Equations

Falling Body: The model for a falling body where air resistance is proportional to the velocity is $\frac{dv}{dt}=g-\frac{k}{m}v$.

Logistic Equation: When the rate of growth of a population P is proportional to the amount present and the amount remaining before reaching the carrying capacity L, then the resulting DE is $\frac{dP}{dt}=kP(L-P)$.

Kirchhoff's Laws

Let $E(t)$ be impressed voltage, $i(t)$ be current, $q(t)$ be charge, L be inductance, R be resistance, and C be capacitance. Current and charge related by $i(t)=\frac{dq}{dt}$.

Conservation of Charge (1st law): The sum of the currents entering a node must equal the sum of the currents exiting a node.

Conservation of Energy (2nd law): The voltages around a closed path in a circuit must sum to zero (voltage drops are negative, voltage gains are positive).

The voltage drop across an inductor is $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$. The voltage drop across a resistor is $iR = R\frac{dq}{dt}$. The voltage drop across a capacitor is $\frac{1}{C}q$. The sum of the voltage drops is equal to the impressed voltage $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$.

Chapter 5 - Modeling with Higher-Order Differential Equations

Spring/Mass Systems

Let $x = 0$ be the equilibrium position with downward being the positive direction.

The main equation is $m\ddot{x} + \beta\dot{x} + kx = f(t)$. Dividing through by the mass, m, and making some substitutions gives us $\ddot{x} + 2\lambda\dot{x} + \omega^2 x = F(t)$, where $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$, and $F(t) = \frac{f(t)}{m}$.

This is a second order linear differential equation with constant coefficients and can be solved using the techniques from chapter 4.

Free motion means that no outside forces act on the system, so $f(t) = 0$ and you have a homogeneous equation. A **driven** system means that an external force acts on the system and you have $f(t) \neq 0$, which makes it a non-homogeneous equation.

An **undamped** system means that there are no retarding forces acting on the spring. In this case the first order term disappears since $\beta = 0$. A **damped** system the medium through which the spring moves slows it down and $\beta > 0$.

We can use $d = \lambda^2 - \omega^2$ to determine the type of damping in the system. A system is overdamped when $d > 0$, underdamped when $d < 0$, and critically damped when $d = 0$. If this sounds like determining the types of roots of a quadratic equation, that's because it is. The discriminant $b^2 - 4ac$ from the general quadratic equation becomes $2d$ in these systems.

Series Circuit Analogue

Kirchhoff's second law $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ is a second order, non-homogeneous, linear differential equation with constant coefficients. It is overdamped, critically damped, or underdamped depending on the value of the discriminant $R^2 - \frac{4L}{C}$.

Deflection of a Beam

Deflection of a horizontal beam satisfies the equation $EI\frac{d^4y}{dx^4} = w(x)$ where EI is the flexural rigidity and $w(x)$ is the load per unit length.

- An embedded end has no deflection, $y = 0$, and is horizontal, $y' = 0$.
- A free end has no bend, $y'' = 0$, and cannot be sheared, $y''' = 0$.
- A simply supported end has no deflection, $y = 0$, and no bend, $y'' = 0$.

Chapter 6 - Series Solutions of Linear Equations

If $x = x_0$ is an ordinary point, then a power series centered at x_0 is $y = \sum_{k=0}^{\infty} c_k (x - x_0)^k$.

Method of Frobenius

If $x = x_0$ is a regular singular point then there exists at least one solution of the form $y = (x - x_0)^r \sum_{k=0}^{\infty} c_k (x - x_0)^k$, which simplifies to $y = \sum_{k=0}^{\infty} c_k (x - x_0)^{k+r}$, where r is a constant to be determined.

Bessel's Equation of Order v

Bessel's equation is $x^2 y'' + xy' + (\alpha^2 x^2 - v^2) y = 0$. The solution is $y = c_1 J_v(\alpha x) + c_2 J_{-v}(\alpha x)$ as long as v is not an integer. If v is integer then the solution is $y = c_1 J_v(\alpha x) + c_2 Y_v(\alpha x)$. Technically, this is a solution to any Bessel equation, but we prefer J_v and J_{-v} when v is not an integer.

The **Modified Bessel Equation** is $x^2 y'' + xy' - (\alpha^2 x^2 + v^2) y = 0$. The solution is similar to the Bessel Equation, except you replace J by I and Y by K.

Legendre's Equation of Order n

Legendre's equation is $(1 - x^2) y'' - 2xy' + n(n+1) y = 0$. If n is a non-negative integer, then the solution is a polynomial $P_n(x)$. The first few solutions are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$. If n is not a non-negative integer, then the solution is an infinite series.

Chapter 7 - The Laplace Transform

> You will have trouble getting the $\mathscr{L}$ symbol, so you may use $\mathcal{L}$ instead. In Microsoft Word's equation editor, you can 1) type `\scriptL` or 2) change the symbols gallery to *Scripts* and click on the $\mathcal{L}$ symbol.

Let f be a function defined for $t \geq 0$. The Laplace transform of $f(t)$ is $F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$, provided this integral converges.

Basic Tranforms

Here are the basic Laplace transformations.

- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
- $\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$
- $\mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

Solving Initial Value Problems

Laplace transforms can be used to solve initial value problems about $t = 0$. One of the huge benefits over what we did in chapter 4 is that this solves the equation and finds the constants at the same time.

1. Take the Laplace transform, $\mathscr{L}$, of both sides

2. Solve for $Y(s)$

3. Take the inverse Laplace transform, $\mathscr{L}^{-1}$, of both sides

Combinations of Functions

There are three types of functions we deal with: polynomial, exponential, and trigonometric functions. If your expression involves only one of the three, then use the basic transform for that function.

If your function contains an exponential and one of the other two types, then find the transform of the other type and apply the translation theorem $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$. For example, in $\mathscr{L}\left\{e^{3t}\cos 4t\right\}$, you would begin with $F(s) = \mathscr{L}\{\cos 4t\} = \frac{s}{s^2+16}$ and then find $F(s-3) = \frac{s-3}{(s-3)^2+16}$.

If you are unable to find a transform of an expression involving a polynomial by one of the other means, then you can use the translation theorem $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

Partial Fractions

You will need to use partial fractions for many of the problems in this chapter. When the denominators are composed of distinct linear factors, you can use the cover-up method. In this method, you cover up a factor and then substitute the corresponding root into the rest of the expression to get the coefficient over the factor.

This shortcut does not work for repeated roots or quadratic factors. In those cases, you may need to use undetermined coefficients to find the values.

Chapter 8 - Systems of Linear First-Order DEs

Eigenvalues and Eigenvectors

If $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is a homogeneous linear first-order system, then the polynomial equation is the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and its solutions are the eigenvalues. We want to write a solution as $\mathbf{X} = \mathbf{K}e^{\lambda t}$, where $\mathbf{K}$ is the associated eigenvector.

The general solution to a homogeneous linear system is $\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}$

If $\mathbf{K}$ is an eigenvector corresponding to the complex eigenvalue $\lambda = \alpha + \beta i$, then let $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K})$ and $\mathbf{B}_2 = \mathrm{Im}(\mathbf{K})$. The two solutions with real coefficients are $\mathbf{X}_1 = [\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t]e^{\alpha t}$ and $\mathbf{X}_2 = [\mathbf{B}_1 \sin\beta t + \mathbf{B}_2 \cos\beta t]e^{\alpha t}$.

For a nonhomogeneous system, the general solution becomes $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ and the method of undetermined coefficients or variation of parameters can be used to find the particular solution.

Matrix Exponentials

For a homogeneous system, we can define a matrix exponential $e^{\mathbf{A}t}$ so that $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution to $\mathbf{X}' = \mathbf{A}\mathbf{X}$. For any square matrix of size n, $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\left(\frac{t^2}{2!}\right) + \mathbf{A}^3\left(\frac{t^3}{3!}\right) + \cdots$, which can be written as $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \left(\frac{t^k}{k!}\right)$.

$\Phi(t) = e^{\mathbf{A}t}$ is a fundamental matrix.

For nonhomogeneous systems, $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, the general solution is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$ where $\mathbf{X}_c = \phi(t)\mathbf{C}$ and $\mathbf{X}_p = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)\,ds$. In practice, $\Phi^{-1}(s) = e^{-\mathbf{A}s}$ can be found from $e^{\mathbf{A}t}$ by substituting $t = -s$.

Chapter 9 - Numerical Solutions of Ordinary DEs

Euler's Method

In chapter 2 (and in Calculus II), we had Euler's Method, which used a sequence of local linear approximations, $f(x+\Delta x) \approx f(x) + \Delta y$ where Δy was approximated by $dy = f'(x)\Delta x$, to estimate the next value in y. Replacing Δx by h and switching to the multivariable form $\frac{dy}{dx} = f(x,y)$, we get the formula used.

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Improved Euler's Method

This method estimates the next y value in the sequence using Euler's method, $y^*_{n+1} = y_n + h \cdot f(x_n, y_n)$, and then averages the slopes at the two ends of the interval to find a better approximation for the next y.

$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$

Runge-Kutta Methods

These are generalizations of Euler's method where the slope $f(x_n, y_n)$ is replaced by a weighted average of the slopes on the interval $x_n \le x \le x_{n+1}$. That is, $y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + \cdots + w_mk_m)$, where the weights w are chosen so that they agree with a Taylor series of order m.

RK1

The first-order Runge-Kutta method is actually Euler's method. Choose $k_1 = f(x_n, y_n)$ and $w_1 = 1$ to get $y_{n+1} = y_n + hk_1$.

RK2

The second-order Runge-Kutta method chooses $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + h, y_n + hk_1)$, and $w_1 = w_2 = \frac{1}{2}$ to get the improved Euler's method where $y_{n+1} = y_n + \frac{1}{2}h[k_1 + k_2]$

RK4

The fourth-order Runge-Kutta method chooses $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$, $k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$, and $k_4 = f(x_n + h, y_n + hk_3)$. The weights are $w_1 = w_4 = \frac{1}{6}$ and $w_2 = w_3 = \frac{1}{3}$. The result is $y_{n+1} = y_n + \frac{1}{6}h[k_1 + 2k_2 + 2k_3 + k_4]$.