Math 160 - Mathematical Notation

Purpose

One goal in any course is to properly use the language of that subject. Finite Mathematics is no different and may often seem like a foreign language. These notations summarize some of the major concepts and more difficult topics of the unit. Typing them helps you learn the material while teaching you to properly express mathematics on the computer. Part of your grade is for properly using mathematical content.

This is not a group assignment, each person needs to create and submit their own notation. If it is determined that sharing occurs, a reduced grade will result for all parties involved.

Instructions

Use Microsoft Word to recreate the following documents and submit them through the Canvas learning management system. Save the Word documents as .docx format; do not send .rtf files.

There will be one document for each chapter. The due dates are listed in Canvas, but you may turn them in early if you like. If you're going to turn in the notations early, it is recommended that you turn one in, get the grade and possible feedback about improving, before you do all of the work. Late work will be accepted but will receive a 20% per day penalty. No late work will be accepted after the final.

Place your name in the heading of the document so that it appears on each page. Failure to put your name in the document will cost you points. Include the chapter and section titles as part of what you type.

You do not have to exactly match the formatting (line breaks, font sizes, colors, etc.,) that is used in this document.

Non-mathematical content should be typed as normal. When you come to mathematical content, you should use the equation editor in Word by going select the Insert toolbar from the menu and clicking on Equation (the π symbol) or using the keyboard shortcut `Alt=`.

The equation editor in Microsoft Word changed with Office 2007 and is friendlier for the masses but not as powerful as the older versions. If you have it available, you may want to use Insert $\rightarrow$ Object $\rightarrow$ Microsoft Equation 3.0 instead (*not available with Mac Office 2016+*), especially for matrices. Another option is to install the free 30-day trial of MathType from Design Science and use it. After 30 days, MathType reverts to a lite mode that still does everything we need it to do.

If there is an equation, put both sides of the equation into the same equation editor box instead of creating two objects. Be sure to use the proper symbols, there are some instances where more than one symbol may look the same, but they have different meanings and don't appear the same as what's on the assignment.

When parentheses are necessary, use balanced parentheses from the toolbar so fractions look like $\left(\frac{1}{3}\right)$ rather than like $(\frac{1}{3})$.

There are a lot of matrices and systems of equations in this course. Matrices, especially those with partition lines in them, are easier to create if you use Insert $\rightarrow$ Object $\rightarrow$ Microsoft Equation 3.0 rather than the built-in equation editor via Insert $\rightarrow$ Equation. Many of the systems of equations have been placed inside of a matrix container for spacing purposes.

There are additional tips on the instructor's website at https://people.richland.edu/james/editor/

Chapter 3 - Finance

Variables representing monetary amounts are in capital letters.

Simple Interest

One payment, interest is not compounded

$$I = Prt$$

I = Interest, P = Principal, r = Rate, t = Time

Compound Interest

One payment, interest is compounded

$$A = P(1+i)^n$$

A = Amount, P = Principal, i = Periodic Rate, n = Number of periods

$$r_e = (1+i)^m - 1$$

r_e = Effective Rate, i = Periodic Rate, m = number of periods per year

Future Value Annuities

A series of payments where the balance grows in value over time

$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

FV = Future Value, PMT = Payment, i = Periodic Rate, n = Number of periods

Present Value Annuities

A series of payments where the balance decreases in value over time

$$PMT = PV\left(\frac{i}{1-(1+i)^{-n}}\right)$$

PV = Present Value, PMT = Payment, i = Periodic Rate, n = Number of periods

Chapter 4 - Systems of Equations, Matrices

Row Operations

These are operations that produce row equivalent matrices:

1. Switch two rows of a matrix: $R_1 \leftrightarrow R_2$
2. Multiply a row by a non-zero constant: $5R_1 \rightarrow R_1$
3. Add a constant multiple of one row to another row: $5R_1 + R_2 \rightarrow R_2$

Reduced Row-Echelon Form (RREF)

A matrix is in reduced row-echelon form when:

1. Rows of all zeros, if there are any, are at the bottom.
2. The first element in any row that is not a zero is a one. This is called the leading one.
3. All elements above and below a leading one are zero. The column with the leading one is called a cleared column.

Reading a solution from an augmented matrix

$$\left[\begin{array}{ccr|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{rcl} x_1 - 2x_3 & = & 5 \\ x_2 + 5x_3 & = & 4 \\ 0 & = & 0 \end{array} \Longrightarrow \begin{array}{rcl} x_1 & = & 5+2t \\ x_2 & = & 4-5t \\ x_3 & = & t \end{array}$$

Matrix Multiplication

Multiply the rows from the first matrix by the columns from the second matrix and add their products together.

$$\begin{bmatrix} 1 & -2 & 5 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -3 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 37 & 3 \\ -17 & 11 \end{bmatrix}$$

In the example shown, $R_1 \times C_1 = 1(6) + (-2)(-3) + 5(5) = 6 + 6 + 25 = 37$.

Leontief Input-Output Model

$$\mathbf{X} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{D}$$

X = Output Matrix, **M** = Technology Matrix, **D** = Demand Matrix

Chapter 5 - Linear Inequalities

System of Linear Inequalities

$$\begin{array}{rcrcr} 2x & + & 5y & \leq & 30 \\ 4x & + & 3y & \leq & 20 \\ x & & & \geq & 0 \\ & & y & \geq & 0 \end{array}$$

Definitions

- The **feasible region** is the region where all of the contraints are satisifed.
- A **corner point** is a vertex of the feasible region.
- A **bounded** feasible region can be enclosed in a circle.
- An **unbounded** feasible region cannot be enclosed in a circle.

Existence of Solutions to a Linear Programming Problem

- For a bounded feasible region, the objective function will always have both a maximum and minimum value of the objective function.
- For an unbounded feasible region with positive coefficients of the objective function, there will be a minimum but no maximum value.
- If the feasible region is empty, then the objective function has no maximum or minimum value.

Fundamental Theorem of Linear Programming

If there is a solution to a linear programming problem, then it will occur at one or more corner points of the feasible region or on the boundary between two corner points.

Non-Negativity Constraints

Although usually not written in the problem itself, almost every story problem has non-negativity constraints. These state that the variables cannot be negative and are written as $x \geq 0$ and $y \geq 0$. These non-negativity constraints limit us to the first quadrant.

Chapter 6 - Linear Programming

Standard Problems

A **standard maximization problem** requires all problem constraints to be in the form of $\leq$ a non-negative constant, but the objective function coefficients can be any real number.

A **standard minimization problem** requires all problem constraints to be in the form of $\geq$ any real number, but the objective function coefficients cannot be negative. The technique for solving a standard minimization problem is to form the dual problem, which is a standard maximization problem, use the Simplex technique, and read the solution from the bottom row.

Standard Maximization Problem

Maximize $P = 300x_1 + 200x_2 + 400x_3$

Subject to

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leq & 200 \\ 5x_1 & + & 2x_2 & + & 4x_3 & \leq & 300 \\ x_1 & + & 4x_2 & + & 4x_3 & \leq & 150 \\ x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

Standard Maximization Initial System

The initial system is what you get after adding slack variables.

Maximize $P = 300x_1 + 200x_2 + 400x_3$

Subject to

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & s_1 & & & & & = & 200 \\ 5x_1 & + & 2x_2 & + & 4x_3 & & & + & s_2 & & & = & 300 \\ x_1 & + & 4x_2 & + & 4x_3 & & & & & + & s_3 & = & 150 \\ x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0 \end{array}$$

Standard Maximization Initial Tableau

This is what you get after moving the objective function variables to the left-hand side and putting it into matrix form.

$$\left[\begin{array}{rrrrrrr|r} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 200 \\ 5 & 2 & 4 & 0 & 1 & 0 & 0 & 300 \\ 1 & 4 & 4 & 0 & 0 & 1 & 0 & 150 \\ \hline -300 & -200 & -400 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Non-Standard Problems

For non-standard maximization problems, do the following

- For each $\leq$ constraint, add a slack variable to the constraint.
- For each $\geq$ constraint, subtract a surplus variable from the constraint, add an artificial variable to the constraint, and subtract M times the artificial variable from the objective function.
- For each $=$ constraint, add an artificial variable to the constraint and subtract M times the artificial variable from the objective function.

Chapter 7 - Logic, Sets, and Counting

Logic

- A logical **and** $p \wedge q$ is true only when both statements are true.
- A logical **or** $p \vee q$ is false only when both statements are false.
- A logical **implies** $p \rightarrow q$ is false only when p is true and q is false.

Sets

- The **union** $A \cup B$ of two sets contains anything found in either set.
- The **intersection** $A \cap B$ of two sets contain only those items found in both sets.

Counting

Fundamental Counting Principle

The total number of ways that two events can happen is found by multiplying together the number of ways that each event can happen.

Factorial

For positive integers, a factorial represents the number of ways you can arrange that many objects without repetition but with regard to order. Written using an exclamation point, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

Combination

If you take a group of n items and divide them into two groups of size r and $n-r$, then the number of ways you can arrange the groups is

$$\frac{n!}{r!(n-r)!}$$

This is called a combination and can be written as can be written as $\binom{n}{r}$ or ${}_nC_r$.

Partitioning

The idea of a combination can be extended to more than two groups. The important part is that you account for every item, which means that values on the bottom add to the value on the top.

For example, the number of ways you can divide 15 people into teams of 5, 4, 3, and 3 people each is

$$\frac{15!}{5! \times 4! \times 3! \times 3!} = 12,612,600$$

Chapter 8 - Probability

Probability Rules

The following assume the universal set is U.

1. All probabilities are between 0 and 1, inclusive. $0 \leq P(A) \leq 1$
2. The sum of the probabilities of all the different outcomes is 1. $\sum P(A_i) = 1$
3. The probability of something that is impossible is 0. $P(\emptyset) = 0$
4. The probability of something that is certain is 1. $P(U) = 1$
5. The probability of something **not** happening is one minus the probability of it happening. $P(A') = 1 - P(A)$
6. The probability of **any** one of several disjoint items happening is the sum of their probabilities. $P(A \cup B) = P(A) + P(B)$
   If the items are not disjoint, then you need to subtract their overlap. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
7. The probability of **all** of several independent items happening is the product of their probabilities. $P(A \cap B) = P(A)P(B)$. If the events are not independent then you need to use conditional probabilities. $P(A \cap B) = P(A)P(B|A)$

Expected Value

The expected value of a probability distribution is also called its mean or average. It is found by multiplying the values by their associated probabilities and then adding those products together.

$$E(x) = \sum xp$$

Decision Theory

Expected value (Bayesian) criterion

This is the only criterion here that uses probabilities. It is useful when repeating a process many times and is the criterion that would be chosen by the realist. You find the expected value under each action and choose the action with the largest expected value.

Maximax criterion

This is the best of the best case scenarios, the criterion an optimist would choose. You find the maximum payoff under each action and then choose the action with the largest best case scenario.

Maximin criterion

This is the best of the worst case scenarios, the criterion a pessimist would choose. Find the minimum payoff under each action and then choose the action with the largest worst case scenario.

Minimax criterion

This method finds the regret or opportunistic loss for each state of nature. These aren't real losses, they are perceived losses. This method uses losses, so smaller is better. The opportunist chooses the best of the worst losses for each action. You first find the opportunistic loss for each state of nature and then find the maximum opportunistic loss for each action and choose the action with the smallest (minimum) loss.

Chapter 9 - Markov Chains

S is a state matrix; they consist of a single row and are subscripted to indicate the number of transitions that have occurred. **P** is the transition matrix; it is a square matrix and represents the probability of going from the row state to the column state. Both are probability matrices, meaning the sum of each row is 1.

Regular Markov Chains

$\mathbf{S}_0$ is the initial state matrix and it represents the beginning conditions. The n^{th}-state matrix is $\mathbf{S}_n = \mathbf{S}_0 P^n$.

Steady State Matrix / Stationary Matrix

The steady state matrix, **S**, represents the long term probability of being in each state. Once you reach the steady state matrix, additional transitions do not change the probabilities, $\mathbf{S} = \mathbf{SP}$.

An easier way to find the steady state matrix exists when the Markov chain is regular. In this case, there is a limiting matrix $\overline{\mathbf{P}} = \mathbf{P}^{\infty}$, where each row is the steady state matrix.

Absorbing Markov Chains

Transition Matrix

The standard form for the transition matrix is to list all absorbing states first and all transient states second. This leads to a natural partitioning, where sub-matrices are defined for each section.

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{R} & \mathbf{Q} \end{array}\right]$$

Fundamental Matrix

The fundamental matrix **F** gives the expected freqencies. Use it whenever a question asks for "how long".

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1}$$

The element in row r, column c of the fundamental matrix represents the expected number of times you will spend in state transient c of the system before ending up at some absorbing state when you start in transient state r.

Long Term Probabilities

The matrix **FR** contains the long term probabilities of ending up in the column state when you start in the row state. The rows are transient states and the columns are all absorbing states since you will eventually end up in an absorbing state in an absorbing Markov chain.

It should be used to answer any questions about "eventual probability" or "long term probability".

FR can also be found from the limiting matrix $\overline{\mathbf{P}}$ where it will be in the lower-left, the transient rows to absorbing columns, section.

$$\overline{\mathbf{P}} = \mathbf{P}^{\infty} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{FR} & \mathbf{0} \end{array}\right]$$

Chapter 10 - Game Theory

2×2 *non-strictly determined game*

Assume $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $D = (a+d) - (b+c)$.

The optimal row player strategy $\mathbf{P}^*$ is

$$\mathbf{P}^* = \begin{bmatrix} \dfrac{d-c}{D} & \dfrac{a-b}{D} \end{bmatrix}$$

The optimal column player strategy $\mathbf{Q}^*$ is

$$\mathbf{Q}^* = \begin{bmatrix} \dfrac{d-b}{D} \\ \dfrac{a-c}{D} \end{bmatrix}$$

The value of the game v is

$$v = \frac{ad-bc}{D}$$

Linear Programming Problem

Assume that $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ has all positive entries. If it doesn't, then add a constant K to all the values first.

The optimal row player strategy $\mathbf{P}^* = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$ is found by solving this standard minimization linear programming problem.

Minimize $z = \frac{1}{v} = x_1 + x_2$, where $x_1 = \frac{p_1}{v}$ and $x_2 = \frac{p_2}{v}$, subject to

$$\begin{array}{rcrcl} ax_1 & + & dx_2 & \geq & 1 \\ bx_1 & + & ex_2 & \geq & 1 \\ cx_1 & + & fx_2 & \geq & 1 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

The optimal column player strategy $\mathbf{Q}^* = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ is found by solving this standard maximization linear programming problem.

Minimize $z = \frac{1}{v} = y_1 + y_2 + y_3$, where $y_1 = \frac{q_1}{v}$, $y_2 = \frac{q_2}{v}$, and $y_3 = \frac{q_3}{v}$, subject to

$$\begin{array}{rcrcrcl} ay_1 & + & by_2 & + & cy_3 & \leq & 1 \\ dy_1 & + & ey_2 & + & fy_3 & \leq & 1 \\ y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

The value of the original game is $v = \frac{1}{z} - K$, where K is the constant you added at the beginning to make the values all positive.